An inverse problem of magnetization in geoscience

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Context

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E. Lima and **B.Weiss**
(Earth, Atmospheric and Planetary Sciences Dpt., MIT, Cambridge Massachusetts, USA)

and **D. Hardin** and **E. Saff**
(Center for Constructive Approximation, Vanderbilt University, Nashville, Tennessee, USA).

- Geophysicists at MIT: study the story of Earth's magnetic field.
  $\rightsquigarrow$ by analysing magnetization characteristics of rocks.
- Not directly observable $\rightsquigarrow$ one observes the induced magnetic field.

Outline

Why study planetary magnetic field?

- Magnetic field is useful:
  - For navigation (compass, migratory birds, some fishes, etc.).
  - It prevents stripping of the atmosphere by the solar wind.
- Complex phenomenon: generated by a "dynamo".
  $\rightsquigarrow$ Several possible mechanisms. Still fairly misunderstood.
- Polarity reversals.
  $\rightsquigarrow$ One of the most convincing evidence of continental drift.

Hot questions:

- Did the moon have a dynamo?
- If so, what was generating it?
- When did it turn on/off?
- same questions for Mars (could explain why Mars lost its atmosphere).

$\rightsquigarrow$ A key question for understanding the early history of the solar system.

How do rocks acquire magnetization?

- Types of rocks: mainly
  - igneous (e.g., from volcanos);
  - or sedimentary (e.g., at the bottom of oceans).
- Thermoremanent magnetization:
  ferro-magnetic particles follow the magnetic field.

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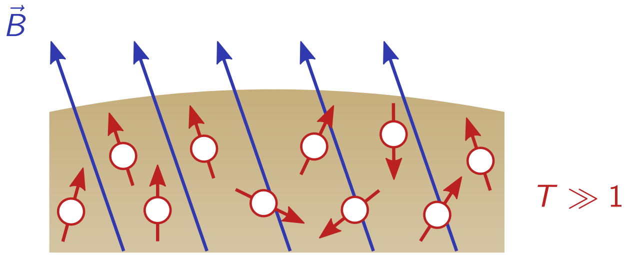

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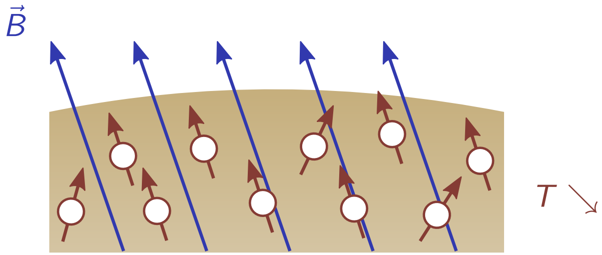

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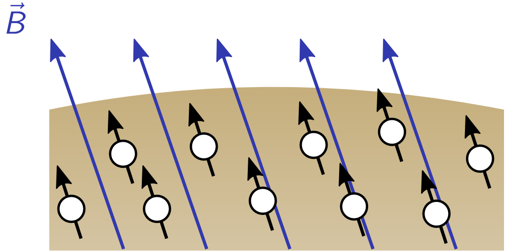

$T \simeq 1$

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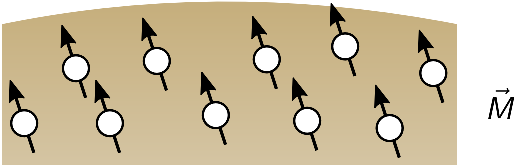

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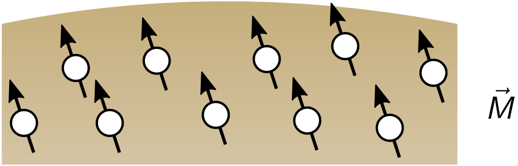

- Can be subsequently altered
  $\rightsquigarrow$ under high pressure or temperature.

Measuring instruments

- Magnetometer: gives the net moment of a sample: $\iiint_{\text{rock}} \vec{M}$.

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SQUID sensors

(Superconducting QUantum Interference Device)

- high sensibility,
- far from the sample (100 μm),
- do not affect the magnetization,
- complicate to operate.

Non-superconducting sensor

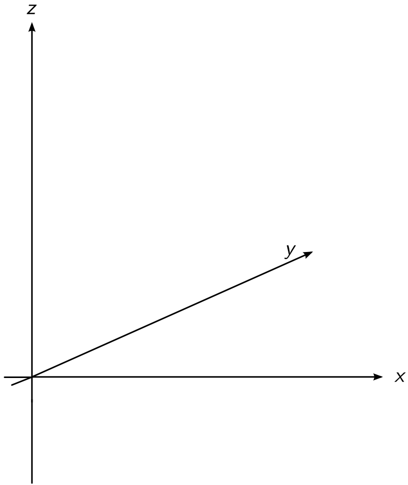
- less sensitive,
- close to the sample (6 μm),
- may induce magnetizations,
- easy to operate.

SQUID microscope

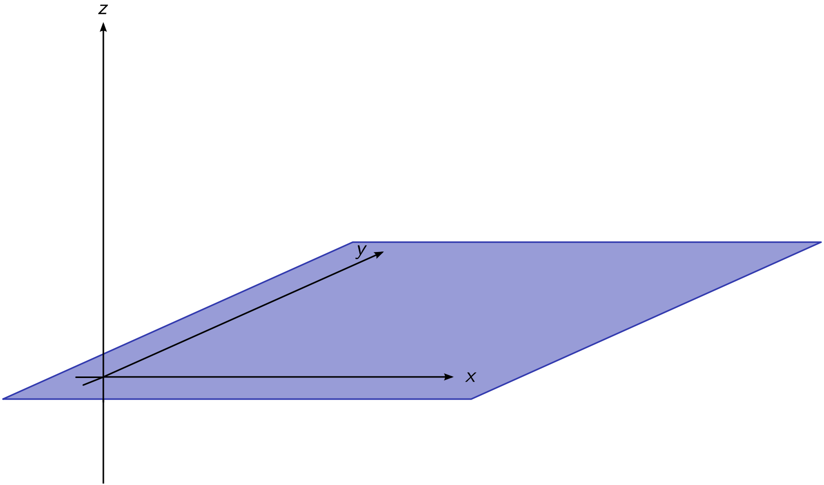
Pedestal + sensor

Sapphire window

General scheme



General scheme



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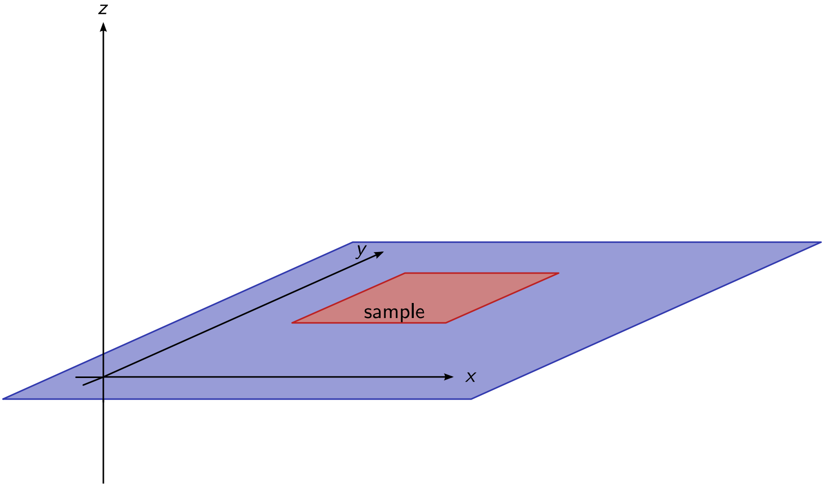

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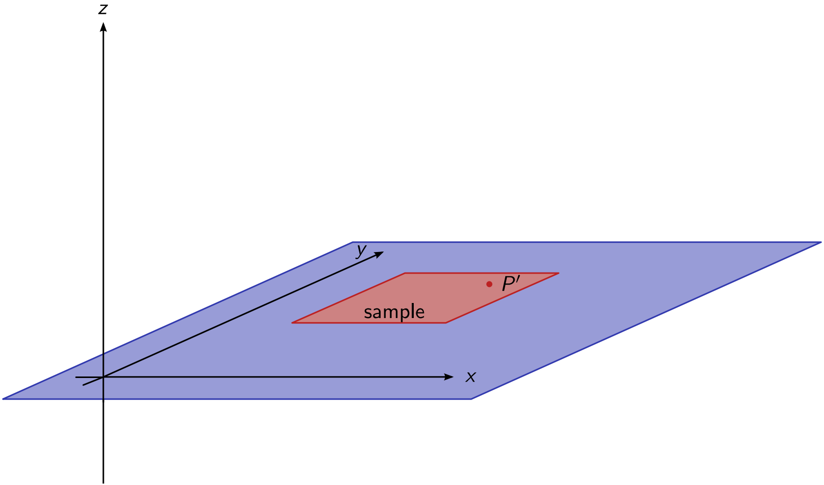

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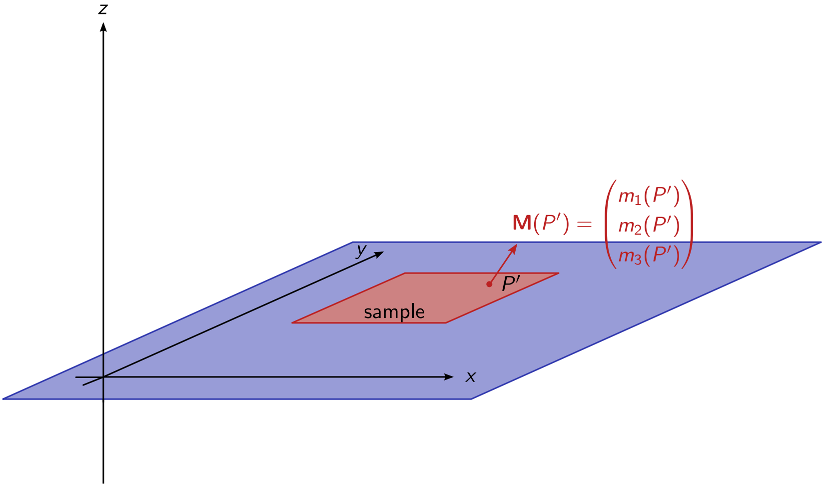

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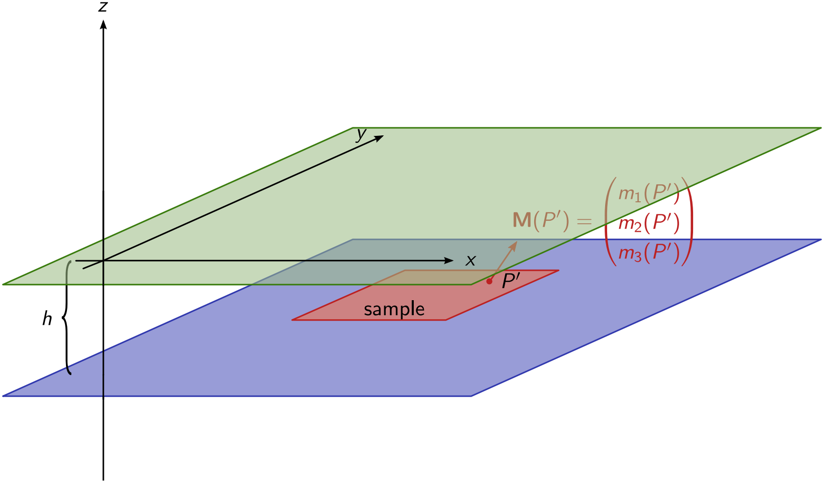

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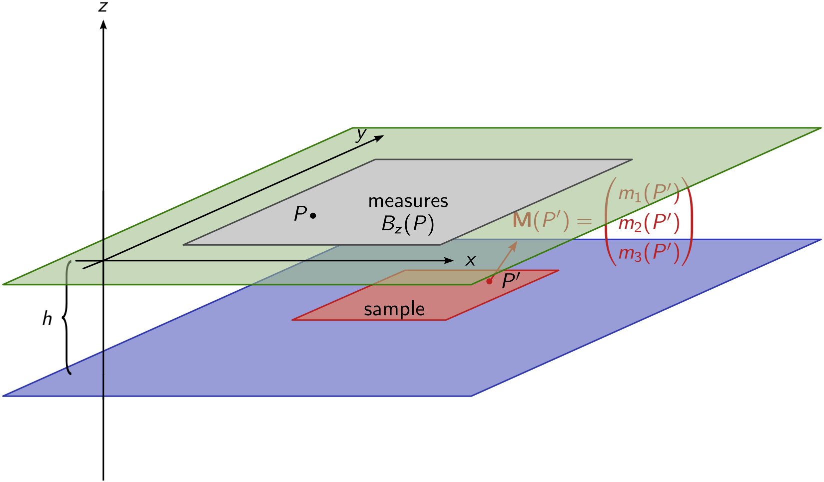


Inverse problem

- Magnetization:

  at each point $P' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ of the sample: $\mathbf{M}(P') = \begin{pmatrix} m_1(P') \\ m_2(P') \\ m_3(P') \end{pmatrix}$.

- Thin-plate hypothesis: $\mathbf{M} \neq 0$ only for $z' = 0$, i.e. $P' = \begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix}$.

- Generates a magnetization potential:

  at any point $P = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ of the space, $\varphi_M(P)$, s.t. $\Delta\varphi_M = \mathrm{div}(\mathbf{M})$.

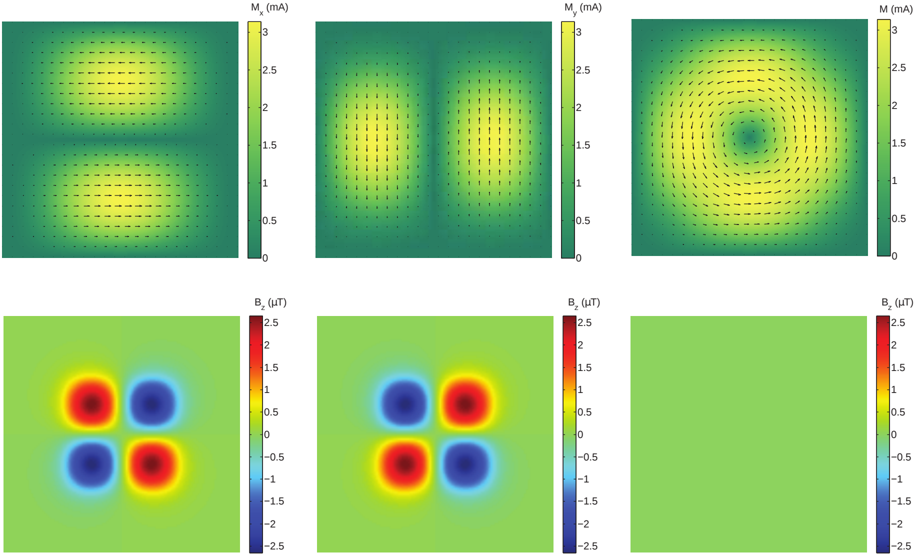
$$\varphi_M(P) = \frac{1}{4\pi} \iint \frac{m_1(P')(x - x') + m_2(P')(y - y')}{|P - P'|^3} + \frac{m_3(P')z}{|P - P'|^3} \mathrm{d}x'\mathrm{d}y'$$

- Magnetic field induced by φ_M: $\mathbf{B}(P) = \nabla\varphi_M(x, y, z)$.

Silent sources

- Inverse problem: from measurements of $\mathbf{B}$ at height h, recover $\mathbf{M}$ (more precisely: only B_z is measured).
- $\mathbf{M}$ is said silent from above (resp. below) if
$$\varphi_M(x, y, z) = 0 \text{ for all } z > 0 \text{ (resp. } z < 0).$$
- Such magnetizations exist. $\rightsquigarrow$ Problem is ill-posed.

A silent source



Regularization hypotheses (1/2)

- Need for further assumptions on $\mathbf{M}$:
  - The support of $\mathbf{M}$ is compact.
- Does compact support help? $\rightsquigarrow$ A bit. In this case:

$$(\mathbf{M} \text{ is silent from above}) \Leftrightarrow (\mathbf{M} \text{ is silent from below}).$$

- There are silent magnetization from both sides: those s.t.

$$m_3 = 0 \text{ and div} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = 0.$$

Regularization hypotheses (2/2)

- **M** can often be supposed unidirectional:

$$\exists\, \mathbf{v} \in \mathbb{R}^3, \exists\, Q : \mathbb{R}^2 \to \mathbb{R} \quad \text{s.t.} \quad \forall P', \mathbf{M}(P') = Q(P')\mathbf{v}.$$

$\rightsquigarrow$ realistic if the rock has not been altered after its formation.

- Does unidirectionality help? $\rightsquigarrow$ No.
  For any **M** and any direction **v** not horizontal, there exists a scalar field Q s.t. $Q(P')\mathbf{v}$ is equivalent from above to **M**.

- If we have compact support **and** unidirectionality? $\rightsquigarrow$ Yes.
  If **M** is unidirectional and compactly supported, there is no unidirectional compactly supported equivalent magnetization.

Outline

Fourier technique

- We consider Fourier transform with respect to horizontal variable $\mathbf{w} = \begin{pmatrix} x \\ y \end{pmatrix}$:

$$\hat{f}(\boldsymbol{\kappa}, z) = \iint f(\mathbf{w}, z) \mathrm{e}^{-2\mathrm{i}\pi(\mathbf{w}\cdot\boldsymbol{\kappa})} \mathrm{d}\mathbf{w}.$$

- For the potential, we get at height $z > 0$:

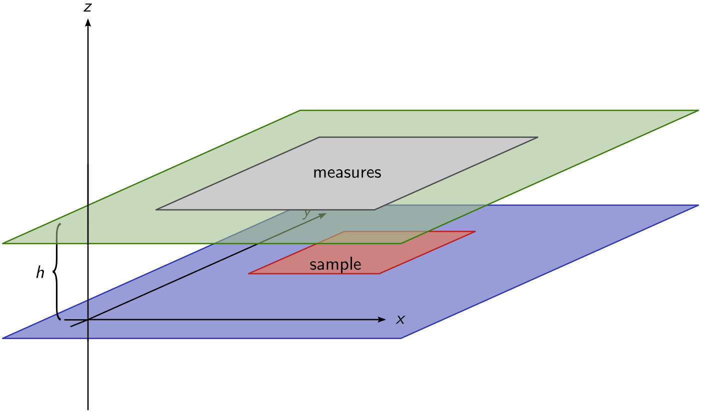
$$\hat{\varphi_M}(\boldsymbol{\kappa}, z) = \frac{\mathrm{e}^{-2\pi z|\boldsymbol{\kappa}|}}{2} \left(\mathrm{i} \frac{\boldsymbol{\kappa}}{|\boldsymbol{\kappa}|} \cdot \begin{pmatrix} \hat{m}_1(\boldsymbol{\kappa}) \\ \hat{m}_2(\boldsymbol{\kappa}) \end{pmatrix} - \hat{m}_3(\boldsymbol{\kappa}) \right).$$

- Case when $\mathbf{M}$ is undirectional: $\mathbf{M}(P') = Q(P')\mathbf{v}$:

$$\hat{\varphi_M}(\boldsymbol{\kappa}, z) = \frac{\mathrm{e}^{-2\pi z|\boldsymbol{\kappa}|}}{2} \left(\mathrm{i} \frac{\boldsymbol{\kappa}}{|\boldsymbol{\kappa}|} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - v_3 \right) \hat{Q}(\boldsymbol{\kappa}).$$

Direct approach

- Discretize the operator $\mathbf{M} \mapsto B_z$:
  - Rectangular regular $n \times n$ magnetization grid.
  - At each node of the grid a dipole.
  - Rectangular regular $N \times N$ measurement grid.



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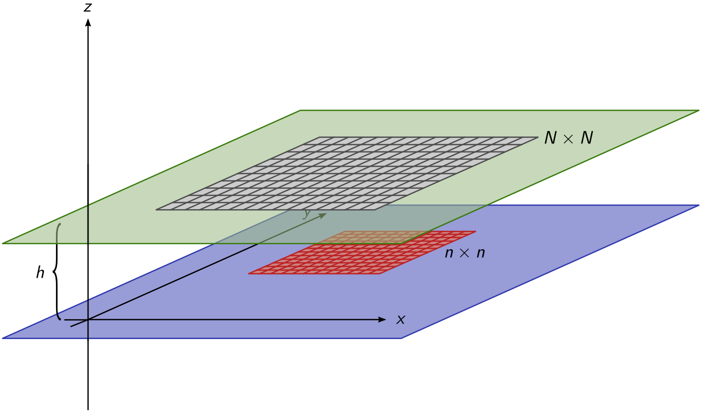

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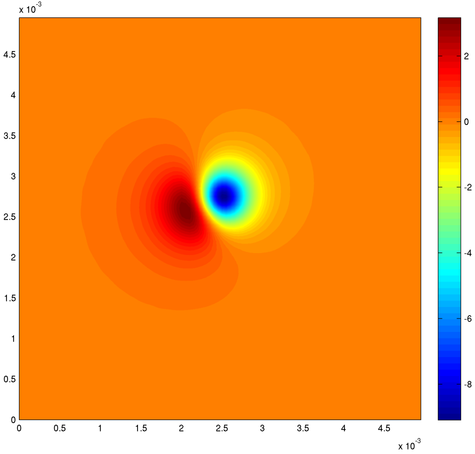
  - Rectangular regular $n \times n$ magnetization grid.
  - At each node of the grid a dipole.
  - Rectangular regular $N \times N$ measurement grid.
- Let A be the matrix of the discrete operator.

  $\rightsquigarrow$ each column: field generated by a single dipole.
- Let b be the vector of measurements. We try to solve $Ax \simeq b$.
- Problem: A very big $\rightsquigarrow$ N^2 rows and $3n^2$ columns

  $\rightsquigarrow$ example: $N = 100$, $n = 50$ $\rightsquigarrow$ 600 MB just to store A.

Lonar spherule example



Field measured for the lonar spherule.

Direct approach (2)

If columns of A are linearly independent:

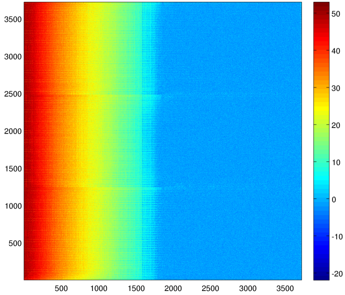
$$(\text{minimize } \|Ax - b\|_2) \quad \Leftrightarrow \quad (\text{solve } A^\star A x = A^\star b).$$

- Interesting if $3n^2 \ll N^2$, i.e. magnetization is localized.
- Computing $A^\star A$ and $A^\star b$: simple dot-products.
- SVD decomposition of a matrix $M = A^\star A$: $M = VSV^\star$ where:
  - S is diagonal and non-negative.
  - V is orthogonal.
- $x = VT$ where $T = S^{-1} V^\star A^\star b$. In other words $x = \sum_k t_k V_k$.
- Solution of min $\|Ax - b\|_2$, subject to $x \in \text{span}(V_1 \ldots V_K)$ is

$$x^{(K)} = \sum_{k=1}^{K} t_k \, V_k.$$

Accuracy issues with the SVD

- Sometimes: accuracy problems with Matlab.
- Criterion: compute `[V S Vtilde] = svd(M);` $\rightsquigarrow$ check $V \simeq \tilde{V}$.

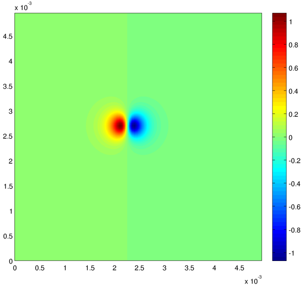


Number of matching bits for each elements of V and $\tilde{V}$

- Other criterion: check that $\|b - \sum_{k=1}^{K} t_k V_k\|$ is decreasing with K.

Using sparse representation?

- Alternative way to save memory consumption: use sparsity.
- Column of A: field generated by a single dipole.
  $\rightsquigarrow$ should decrease quickly.

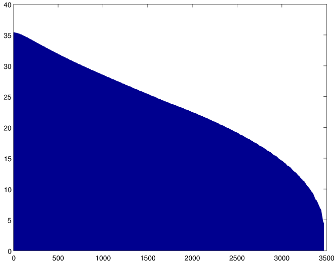


Field produced by a single dipole

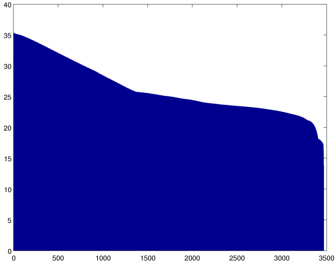
- Approximate A by replacing values smaller than some threshold by 0.

Sparse representation

- Sparse representation of $A \rightsquigarrow$ easily fits in memory.
- Fast computation of $A^\star A$.
- But does not reduce computation time for the SVD.
- And...deeply change the behavior of A even with small threshold.



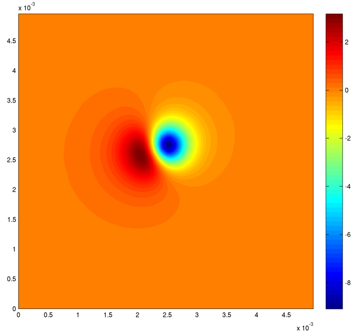
Singular values of $A^\star A$



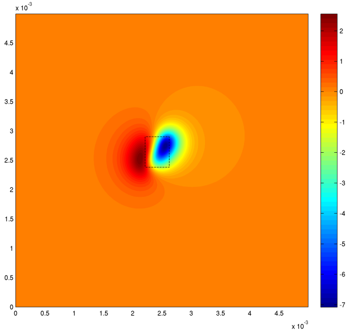
Idem sparse case: threshold $= 0.1\%$

Lonar spherule and synthetic example

- Lonar spherule: very small sphere of rock.
- Synthetic example: looks like the spherule example.
  $\rightsquigarrow$ Allows for benchmarking.



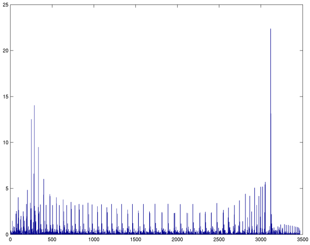
Field measured for the lonar spherule.



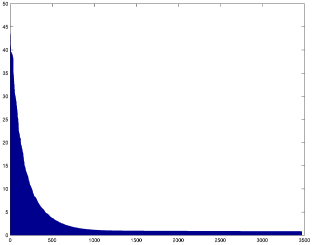
Field of the synthetic example.

Synthetic example, 1st step

- Magnetization grid: 34×34, same dimensions as measurement grid.
- Field very well reconstructed: $\|Ax - b\|_2 / \|b\|_2 \simeq 1.7\%$.
- But last singular vectors have significant moments.



Amplitude of the net moment of V_K



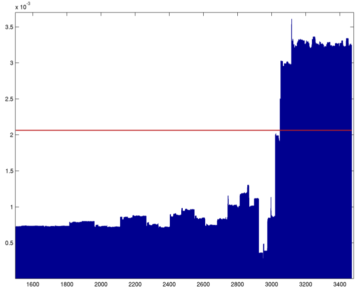
$\|Ax^{(K)} - b\|$.

$$x^{(K)} = \sum_{k=1}^{K} t_k \, V_k.$$

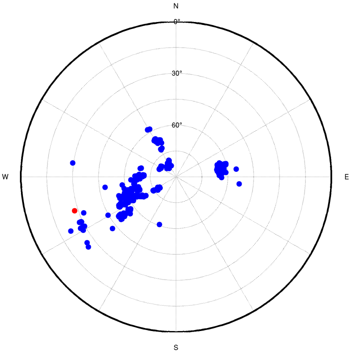
$\rightsquigarrow$ does not permit to deduce the net moment.

1st step: recovered moment

Magnetizations explaining equally well the measured field.
(true moment in red)



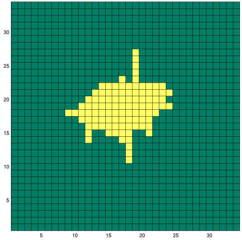
Amplitude of the net moment of $x^{(K)}$.



Direction of the net moment of $x^{(K)}$.

Synthetic example, 2nd step

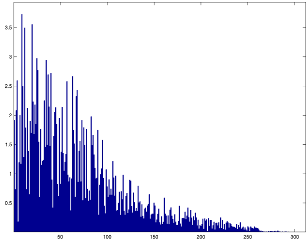
- We expect the support of the magnetisation to be very localised.
- We discard points of the magnetization grid, using a thresholding strategy based on the result of the 1st step.
- Second step with new support.



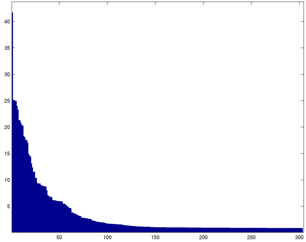
New support in yellow (threshold = 10%)

Synthetic example, 2nd step

- Field still well reconstructed: $\|Ax - b\|_2 / \|b\|_2 \simeq 1.76\%$.
- This time, moments of the last singular vectors are very small.
- Last singular vectors have small moments.



Amplitude of the net moment of V_K

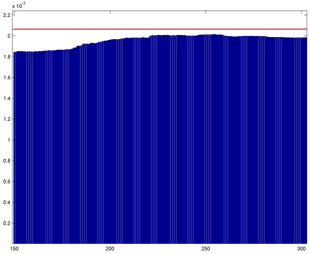


$\|Ax^{(K)} - b\|$.

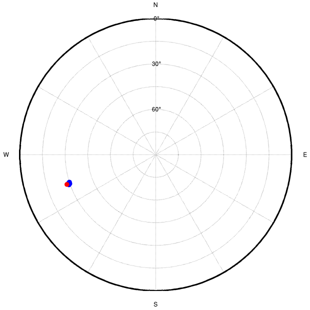
$$x^{(K)} = \sum_{k=1}^{K} t_k \, V_k.$$

2nd step: recovered moment

Net moments of the least-square solution on span$(V_1 \ldots V_K)$
(true moment in red)



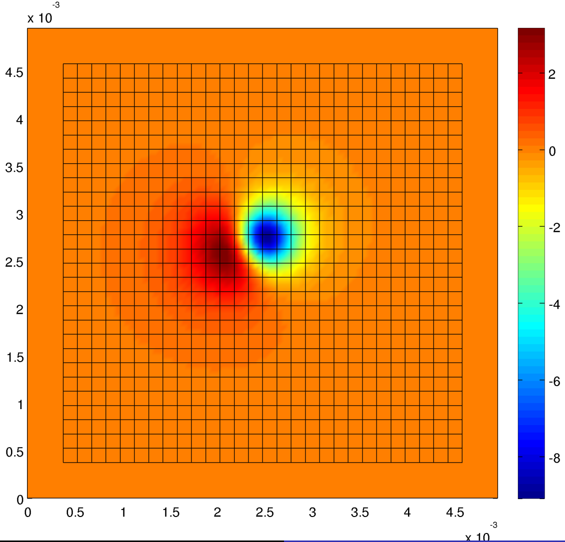
Amplitude of the net moment of $x^{(K)}$.



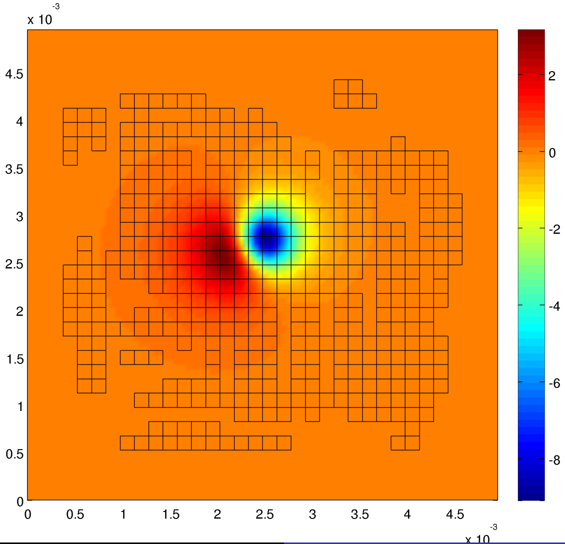
Direction of the net moment of $x^{(K)}$.

- Net moment remarkably well recovered.
  $\rightsquigarrow$ shrinking the support regularizes the problem.

Lonar spherule example



Lonar spherule example



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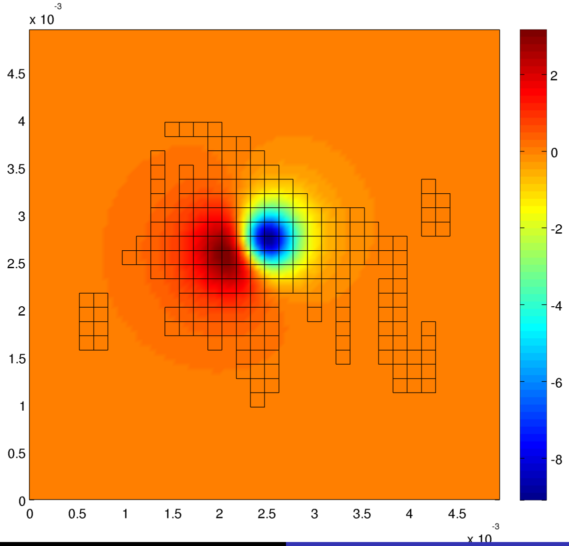

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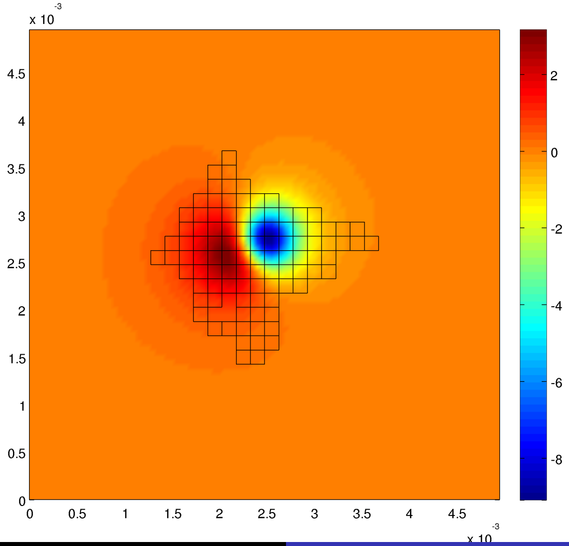

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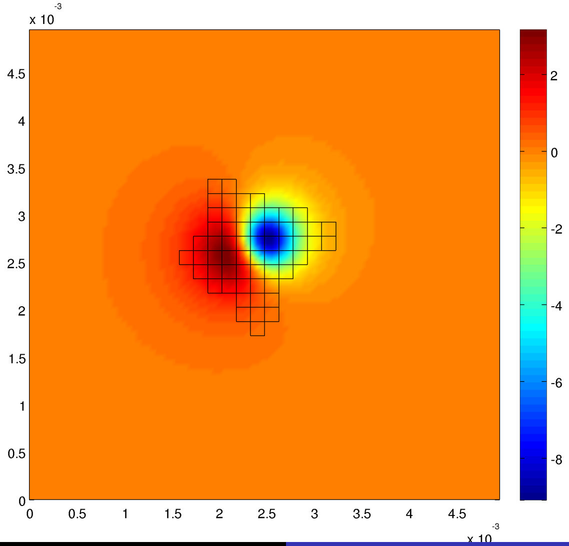

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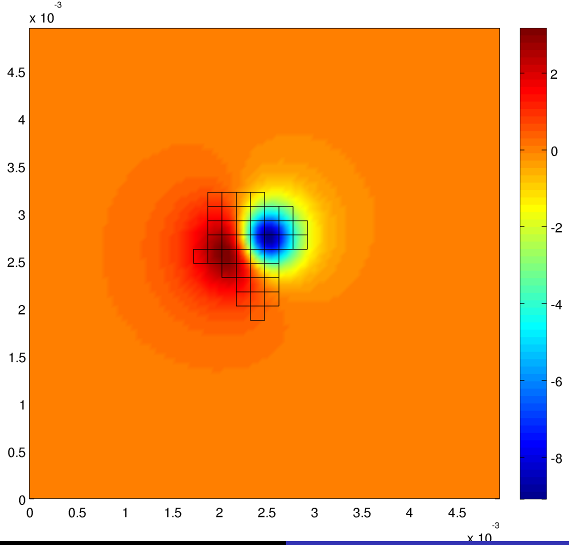
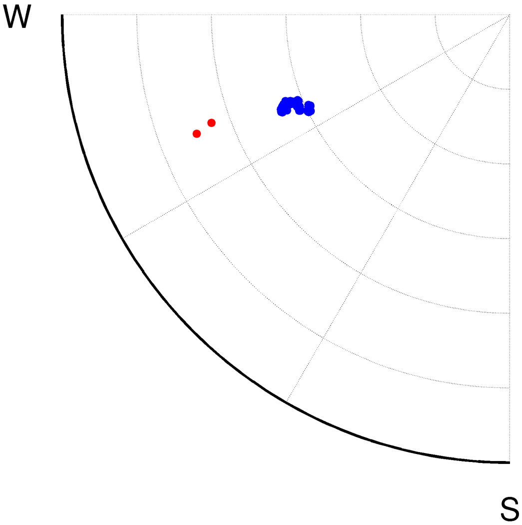


Evolution of the angle of the net moment



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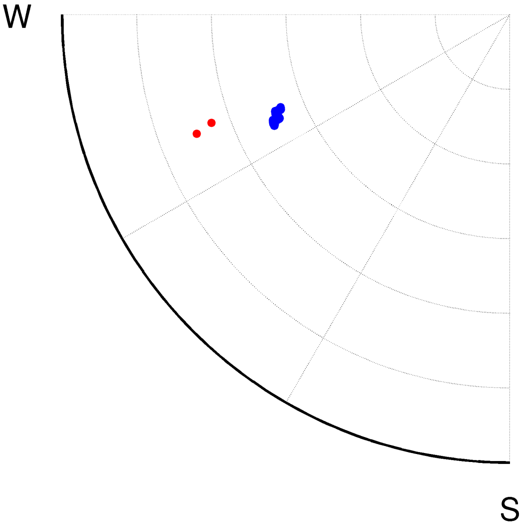

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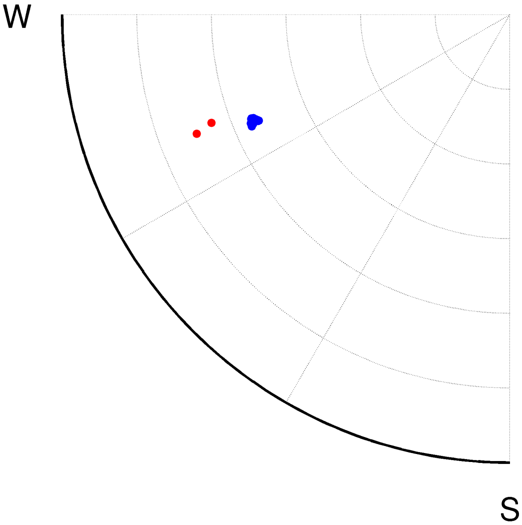

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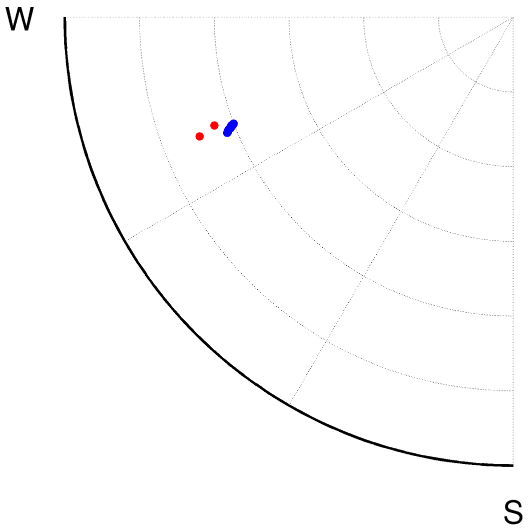

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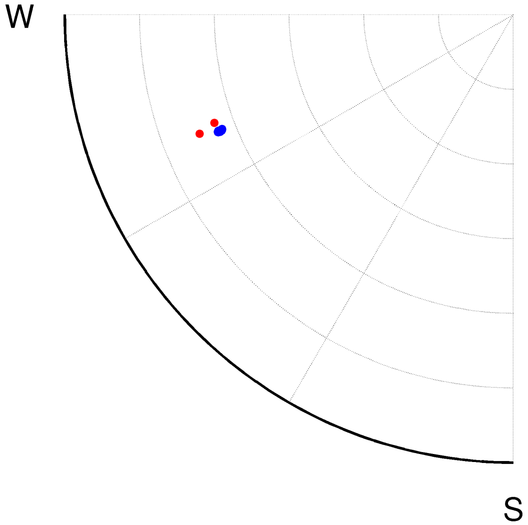
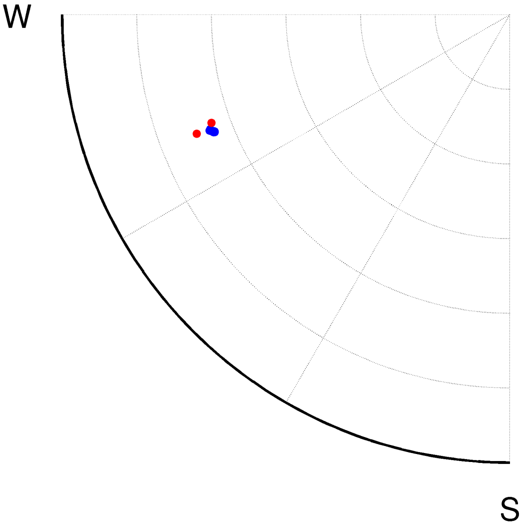


Evolution of the angle of the net moment



Preliminary results

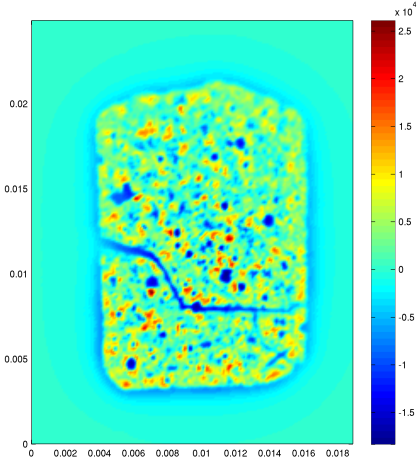
- This strategy in two steps has been used on the lonar spherule example and on some chondrules from Allende meteorite.
- The net moment recovered by this method matches the net moment measured with a magnetometer.

Example	Recovered moment			Measured with magnetometer		
	r	θ	φ	r	θ	φ
Lon. sph.	5 e−6	117.5	−159.2	5.31 e−6	112.7	−159.2
A1b1	1.45 e−9	85.7	11.7	1.54 e−9	87.2	14.9
A1b4	7.4 e−10	160	155	6.28 e−10	155	160
A1b6	1.8 e−11	92	223.8	1.73 e−11	93.2	234.5

Non-localised example

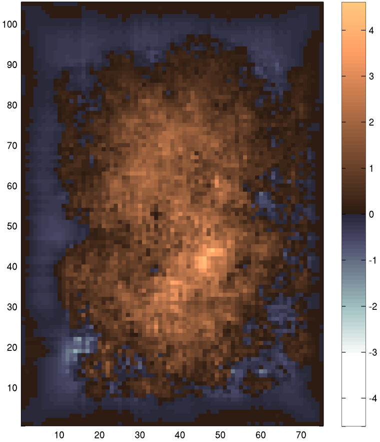
Photograph of an Hawaian basalt

Non-localised example



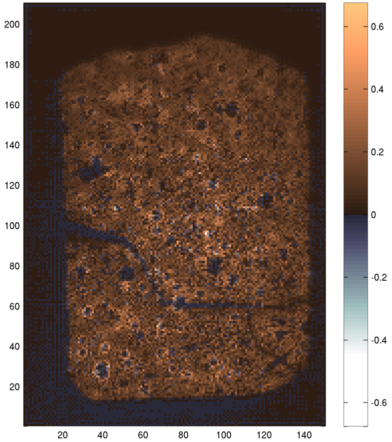
Measured field B_z

Non-localised example



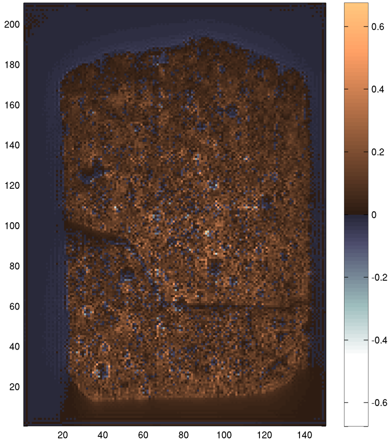
Recovered 3D magnetization (normal component)

Non-localised example



Recovered unidirectional magnetization (right direction)

Non-localised example

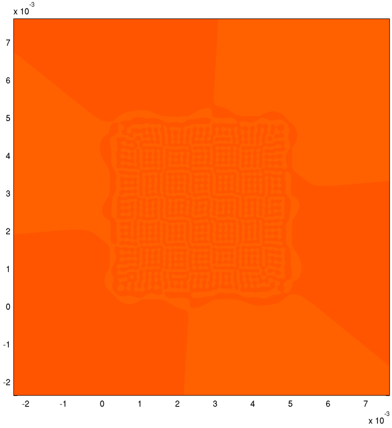


Recovered unidirectional magnetization (wrong direction)

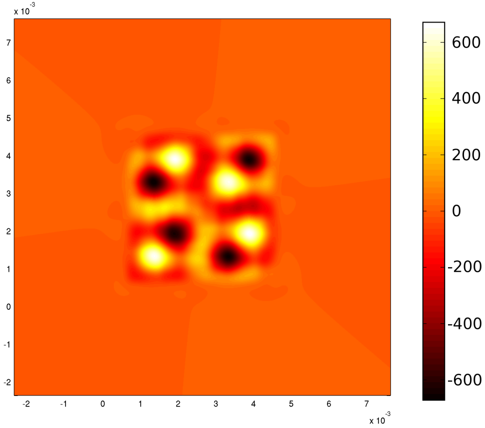
What next?

- Use less naive strategies to compute the net moment of the magnetization.
- Once a plausible magnetization is recovered, find an equivalent unidirectional magnetization.
- Problem: existence of fairly silent magnetization from above, but not from below.
- Solution? Measure from both sides?

Silent source from only one side



B_z measured at height $h = 270\mu$m



B_z measured at height $h = -270\mu$m

Field produced by a 28×28 grid of uniformly magnetized squares.