An inverse problem of magnetization in geoscience

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February 26, 2015
Context

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(Earth, Atmospheric and Planetary Sciences Dpt., MIT, Cambridge Massachusetts, USA)

and D. Hardin and E. Saff
(Center for Constructive Approximation, Vanderbilt University, Nashville, Tennessee, USA).

- Geophysicists at MIT: study the story of Earth’s magnetic field.
  \(\leadsto\) by analysing magnetization characteristics of rocks.

- Not directly observable \(\leadsto\) one observes the induced magnetic field.
Outline

Motivation

Strategies

Preliminary results
Why study planetary magnetic field?

- Magnetic field is useful:
  - For navigation (compass, migratory birds, some fishes, etc.).
  - It prevents stripping of the atmosphere by the solar wind.
- Complex phenomenon: generated by a “dynamo”.
  - Several possible mechanisms. Still fairly misunderstood.
- Polarity reversals.
  - One of the most convincing evidence of continental drift.

Hot questions:

- Did the moon have a dynamo?
- If so, what was generating it?
- When did it turn on/off?
- same questions for Mars (could explain why Mars lost its atmosphere).

⇒ A key question for understanding the early history of the solar system.
How do rocks acquire magnetization?

- Types of rocks: mainly
  - igneous (e.g., from volcanos);
  - or sedimentary (e.g., at the bottom of oceans).
- Thermoremanent magnetization:
  ferro-magnetic particles follow the magnetic field.
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\[ \overrightarrow{B} \]

\[ T \approx 1 \]
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- Thermo-remanent magnetization:
  ferro-magnetic particles follow the magnetic field.

- Can be subsequently altered
  \(\sim\) under high pressure or temperature.
Measuring instruments

- Magnetometer: gives the net moment of a sample: $\iiint_{\text{rock}} \vec{M}$. 
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- Scanning Magnetic Microscopes (SMM):
Measuring instruments

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- Scanning Magnetic Microscopes (SMM):

**SQUID sensors**
(Superconducting QUantum Interference Device)
- high sensibility,
- far from the sample (100 \( \mu \text{m} \)),
- do not affect the magnetization,
- complicate to operate.

**Non-superconducting sensor**
- less sensitive,
- close to the sample (6 \( \mu \text{m} \)),
- may induce magnetizations,
- easy to operate.
SQUID microscope

Pedestal + sensor

Sapphire window
General scheme

\[ x \quad y \quad z \]
General scheme
General scheme
General scheme
General scheme

\[ \mathbf{M}(P') = \begin{pmatrix} m_1(P') \\ m_2(P') \\ m_3(P') \end{pmatrix} \]
General scheme

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General scheme
Inverse problem

- **Magnetization:**
  
  at each point $P' = \left( \begin{array}{c} x' \\ y' \\ z' \end{array} \right)$ of the sample:  
  
  $\mathbf{M}(P') = \left( \begin{array}{c} m_1(P') \\ m_2(P') \\ m_3(P') \end{array} \right)$. 

- **Thin-plate hypothesis:** $\mathbf{M} \neq 0$ only for $z' = 0$, i.e. $P' = \left( \begin{array}{c} x' \\ y' \\ 0 \end{array} \right)$. 

- **Generates a magnetization potential:**
  
  at any point $P = \left( \begin{array}{c} x \\ y \\ z \end{array} \right)$ of the space, $\varphi_M(P)$, s.t. $\Delta \varphi_M = \text{div}(\mathbf{M})$. 

  $\varphi_M(P) = \frac{1}{4\pi} \iint \frac{m_1(P')(x - x') + m_2(P')(y - y')}{|P - P'|^3} + \frac{m_3(P')z}{|P - P'|^3} \, dx' \, dy'$ 

- **Magnetic field induced by $\varphi_M$:** $\mathbf{B}(P) = \nabla \varphi_M(x, y, z)$. 

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Silent sources

- **Inverse problem:** from measurements of $B$ at height $h$, recover $M$ (more precisely: only $B_z$ is measured).

- $M$ is said **silent from above** (resp. below) if
  \[ \varphi_M(x, y, z) = 0 \text{ for all } z > 0 \text{ (resp. } z < 0). \]

- Such magnetizations exist. $\implies$ Problem is ill-posed.
A silent source

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Regularization hypotheses (1/2)

- Need for further assumptions on $\mathbf{M}$:
  - The support of $\mathbf{M}$ is compact.

- Does compact support help? $\rightsquigarrow$ A bit. In this case:
  
  $(\mathbf{M} \text{ is silent from above}) \Leftrightarrow (\mathbf{M} \text{ is silent from below}).$

- There are silent magnetization from both sides: those s.t.
  
  $$m_3 = 0 \text{ and } \text{div} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = 0.$$
Regularization hypotheses (2/2)

- **M** can often be supposed unidirectional:
  \[ \exists \mathbf{v} \in \mathbb{R}^3, \exists Q : \mathbb{R}^2 \to \mathbb{R} \text{ s.t. } \forall P', M(P') = Q(P') \mathbf{v}. \]

→ realistic if the rock has not been altered after its formation.

- Does unidirectionality help? → No.
  For any **M** and any direction **v** not horizontal, there exists a scalar field **Q** s.t. \( Q(P') \mathbf{v} \) is equivalent from above to **M**.

- If we have compact support **and** unidirectionality? → Yes.
  If **M** is unidirectional and compactly supported, there is no unidirectional compactly supported equivalent magnetization.
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Fourier technique

- We consider Fourier transform with respect to horizontal variable $w = \begin{pmatrix} x \\ y \end{pmatrix}$:

$$\hat{f}(\kappa, z) = \int \int f(w, z) e^{-2i\pi (w \cdot \kappa)} dw.$$ 

- For the potential, we get at height $z > 0$:

$$\hat{\varphi}_M(\kappa, z) = e^{-2\pi z |\kappa|} \frac{1}{2} \left( i \frac{\kappa}{|\kappa|} \cdot \left( \hat{m}_1(\kappa) \hat{m}_2(\kappa) \right) - \hat{m}_3(\kappa) \right).$$

- Case when $M$ is undirectional: $M(P') = Q(P')v$:

$$\hat{\varphi}_M(\kappa, z) = e^{-2\pi z |\kappa|} \frac{1}{2} \left( i \frac{\kappa}{|\kappa|} \cdot \left( \begin{array}{c} v_1 \\ v_2 \end{array} \right) - v_3 \right) \hat{Q}(\kappa).$$
Direct approach

- Discretize the operator $\mathbf{M} \mapsto B_z$:
  - Rectangular regular $n \times n$ magnetization grid.
  - At each node of the grid a dipole.
  - Rectangular regular $N \times N$ measurement grid.

![Diagram of magnetization measurement setup](image)

- Let $\mathbf{A}$ be the matrix of the discrete operator.
- Each column: field generated by a single dipole.
- Let $\mathbf{b}$ be the vector of measurements. We try to solve $\mathbf{A}\mathbf{x} \simeq \mathbf{b}$.

- Problem: $\mathbf{A}$ very big $\Rightarrow N^2$ rows and $3n^2$ columns.
  - Example: $N = 100$, $n = 50 \Rightarrow 600$ MB just to store $\mathbf{A}$.
Direct approach

- Discretize the operator \( M \mapsto B_z \):
  - Rectangular regular \( n \times n \) magnetization grid.
  - At each node of the grid a dipole.
  - Rectangular regular \( N \times N \) measurement grid.

\[ A \text{ is the matrix of the discrete operator.} \]
\[ \text{Each column: field generated by a single dipole.} \]
\[ \text{Let } b \text{ be the vector of measurements. We try to solve } Ax \approx b. \]
\[ \text{Problem: } A \text{ very big} \]
\[ \rightarrow \text{ } \text{ } N^2 \text{ rows and } 3n^2 \text{ columns} \]
\[ \rightarrow \text{ example: } N = 100, n = 50 \]
\[ \rightarrow \text{ 600 MB just to store } A. \]
Direct approach

- Discretize the operator $M \mapsto B_z$:
  - Rectangular regular $n \times n$ magnetization grid.
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- Let $A$ be the matrix of the discrete operator.
  \(~\leadsto~\) each column: field generated by a single dipole.

- Let $b$ be the vector of measurements. We try to solve $Ax \simeq b$.

- Problem: $A$ very big $\leadsto N^2$ rows and $3n^2$ columns
  \(~\leadsto~\) example: $N = 100$, $n = 50 \leadsto 600$ MB just to store $A$. 
Lonar spherule example

Field measured for the lonar spherule.
Direct approach (2)

If columns of $A$ are linearly independent:

$$(\text{minimize } \|Ax - b\|_2) \iff (\text{solve } A^*Ax = A^*b).$$

- Interesting if $3n^2 \ll N^2$, i.e. magnetization is localized.
- SVD decomposition of a matrix $M = A^*A$: $M = VS\!V^*$ where:
  - $S$ is diagonal and non-negative.
  - $V$ is orthogonal.
- $x = VT$ where $T = S^{-1}V^*A^*b$. In other words $x = \sum_k t_k V_k$. 
- Solution of $\text{min } \|Ax - b\|_2$, subject to $x \in \text{span}(V_1 \ldots V_K)$ is
  $$x^{(K)} = \sum_{k=1}^K t_k V_k.$$
Accuracy issues with the SVD

- Sometimes: accuracy problems with Matlab.
- Criterion: compute \([V \ S \ Vtilde] = \text{svd}(M)\); \(\sim\) check \(V \simeq \tilde{V}\).

Number of matching bits for each elements of \(V\) and \(\tilde{V}\)

- Other criterion: check that \(\|b - \sum_{k=1}^{K} t_k V_k\|\) is decreasing with \(K\).
Using sparse representation?

- Alternative way to save memory consumption: use sparsity.
- Column of $A$: field generated by a single dipole. 
  $\Rightarrow$ should decrease quickly.

Field produced by a single dipole

- Approximate $A$ by replacing values smaller than some threshold by 0.
Sparse representation

- Sparse representation of $A \leadsto$ easily fits in memory.
- Fast computation of $A^*A$.
- But does not reduce computation time for the SVD.
- And... deeply change the behavior of $A$ even with small threshold.

Singular values of $A^*A$

Idem sparse case: threshold = 0.1%
Lonar spherule and synthetic example

- Lonar spherule: very small sphere of rock.
- Synthetic example: looks like the spherule example.
  \( \leadsto \) Allows for benchmarking.

Field measured for the lonar spherule.

Field of the synthetic example.
Synthetic example, 1st step

- Magnetization grid: $34 \times 34$, same dimensions as measurement grid.
- Field very well reconstructed: $\|Ax - b\|_2 / \|b\|_2 \approx 1.7\%$.
- But last singular vectors have significant moments.

\[
\|Ax^{(K)} - b\|.
\]

\[
x^{(K)} = \sum_{k=1}^{K} t_k V_k.
\]

\[
\Rightarrow \text{does not permit to deduce the net moment.}
\]
1st step: recovered moment

Magnetizations explaining equally well the measured field.

(true moment in red)

Amplitude of the net moment of $x^{(K)}$.

Direction of the net moment of $x^{(K)}$. 
Synthetic example, 2nd step

- We expect the support of the magnetisation to be very localised.
- We discard points of the magnetization grid, using a thresholding strategy based on the result of the 1st step.
- Second step with new support.

New support in yellow (threshold = 10%)
Synthetic example, 2nd step

- Field still well reconstructed: \( \frac{\|Ax - b\|_2}{\|b\|_2} \approx 1.76\% \).
- This time, moments of the last singular vectors are very small.
- Last singular vectors have small moments.

\[
||Ax^{(K)} - b|| = \sum_{k=1}^{K} t_k V_k.
\]
2nd step: recovered moment

Net moments of the least-square solution on \( \text{span}(V_1 \ldots V_K) \)
(true moment in red)

Amplitude of the net moment of \( x^{(K)} \).

Net moment remarkably well recovered.

\( \leadsto \) shrinking the support regularizes the problem.
Lonar spherule example
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Evolution of the angle of the net moment

![Diagram showing the evolution of the angle of the net moment.](image)
Evolution of the angle of the net moment
Evolution of the angle of the net moment

![Graph showing the evolution of the angle of the net moment with points indicating iterations]
Evolution of the angle of the net moment

![Diagram of polar coordinates with arrows and points representing the evolution of the angle of the net moment](image)
Evolution of the angle of the net moment
Evolution of the angle of the net moment

\[ W \quad S \]
Preliminary results

- This strategy in two steps has been used on the lonar spherule example and on some chondrules from Allende meteorite.

- The net moment recovered by this method matches the net moment measured with a magnetometer.

<table>
<thead>
<tr>
<th>Example</th>
<th>Recovered moment</th>
<th>Measured with magnetometer</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( r ) \hspace{1cm} ( \theta ) \hspace{1cm} ( \varphi )</td>
<td>( r ) \hspace{1cm} ( \theta ) \hspace{1cm} ( \varphi )</td>
</tr>
<tr>
<td>Lon. sph.</td>
<td>5 e−6 \hspace{1cm} 117.5 \hspace{1cm} −159.2</td>
<td>5.31 e−6 \hspace{1cm} 112.7 \hspace{1cm} −159.2</td>
</tr>
<tr>
<td>A1b1</td>
<td>1.45 e−9 \hspace{1cm} 85.7 \hspace{1cm} 11.7</td>
<td>1.54 e−9 \hspace{1cm} 87.2 \hspace{1cm} 14.9</td>
</tr>
<tr>
<td>A1b4</td>
<td>7.4 e−10 \hspace{1cm} 160 \hspace{1cm} 155</td>
<td>6.28 e−10 \hspace{1cm} 155 \hspace{1cm} 160</td>
</tr>
<tr>
<td>A1b6</td>
<td>1.8 e−11 \hspace{1cm} 92 \hspace{1cm} 223.8</td>
<td>1.73 e−11 \hspace{1cm} 93.2 \hspace{1cm} 234.5</td>
</tr>
</tbody>
</table>
Non-localised example

Photograph of an Hawaiian basalt
Non-localised example

Measured field $B_z$
Non-localised example

Recovered 3D magnetization (normal component)
Non-localised example

Recovered unidirectional magnetization (right direction)
Non-localised example

Recovered unidirectional magnetization (wrong direction)
What next?

- Use less naive strategies to compute the net moment of the magnetization.

- Once a plausible magnetization is recovered, find an equivalent unidirectional magnetization.

- Problem: existence of fairly silent magnetization from above, but not from below.

- Solution? Measure from both sides?
Silent source from only one side

\( B_z \) measured at height \( h = 270 \mu m \)

\( B_z \) measured at height \( h = -270 \mu m \)

Field produced by a \( 28 \times 28 \) grid of uniformly magnetized squares.