Totality, towards completeness

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Contributions

1. Total finiteness spaces $(A, \mathcal{T}(A))$, a semantics for classical linear logic.
2. Barycentric boolean calculus, a parallel syntax which is total:

$$\begin{aligned} \mathbf{s} \;&::=\; \mathbf{x} \in \mathcal{V} \mid \lambda \mathbf{x}.\mathbf{s} \mid (\mathbf{s})\mathbf{S} \\ &\quad\mid \mathbf{T} \mid \mathbf{F} \mid \texttt{if}\ \mathbf{s}\ \texttt{then}\ \mathbf{R}\ \texttt{else}\ \mathbf{S} \end{aligned}$$

$$\mathbf{R}, \mathbf{S} \;::=\; \sum_{i=1}^{m} a_i\, \mathbf{s}_i \qquad \text{where } \sum_{i=1}^{m} a_i = 1\,.$$

3. Full completeness at the first order boolean type.

Theorem (Completeness)

Every total function of $\mathcal{T}(\mathcal{B}^n \Rightarrow \mathcal{B})$ is the interpretation of a term of the boolean barycentric calculus.

Context

Linear Logic et differential lambda-calculus

80's Linear logic and linear algebra.

2000's Finiteness spaces.

Syntax Differential syntaxes.

Denotational semantics

70's-90's The quest for sequentiality, through the full adequacy issue.

2000's The quest for non-determinism passing by differential lambda-calculus.

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   - Full completeness and non determinism
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What about parallel algorithm

Non-deterministic algorithm are made of different programs of the same type that are reduced in parallel.

We use **algebraic λ-calculus** which is equipped with sums and scalar coefficients which give account of the number of way to compute a result (cf. Boudol, Vaux, Ehrhard-Regnier).

We tackle the **full completeness** question from both traditional viewpoint and

- vary the model to fit a language,
- vary the language to fit the model.

Algebraic λ-Calculus

Simply typed λ-calculus:

$$\frac{}{\Gamma, x : A \vdash x : A}\ (\texttt{var})$$

$$\frac{\Gamma, x : A \vdash \mathbf{s} : B}{\Gamma \vdash \lambda x.\mathbf{s} : A \Rightarrow B}\ (\texttt{abs}) \qquad \frac{\Gamma \vdash \mathbf{s} : A \Rightarrow B \quad \Gamma \vdash \mathbf{r} : A}{\Gamma \vdash (\mathbf{s})\mathbf{r} : B}\ (\texttt{app})$$

Algebraic extension:

$$\frac{}{\Gamma \vdash 0 : A}\ (\texttt{0})$$

$$\frac{\Gamma \vdash \mathbf{s}_1 : A \quad \Gamma \vdash \mathbf{s}_2 : A}{\Gamma \vdash \mathbf{s}_1 + \mathbf{s}_2 : A}\ (\texttt{sum}) \qquad \frac{\Gamma \vdash \mathbf{s} : A \quad a \in \Bbbk}{\Gamma \vdash a\mathbf{s} : A}\ (\texttt{scal})$$

Zero proves any formula, it stands for non *total proof* like the Daimon ✠ of Girard

The barycentric boolean calculus

Let $\mathbb{k}$ be a infinite field and $\mathcal{V}$ be a countable set of variables.

Definition (λ + +)

Atomic terms **s** and barycentric terms **T** are inductively defined

$$\mathbf{s} \quad ::= \quad \mathbf{x} \in \mathcal{V} \mid \lambda \mathbf{x}.\mathbf{s} \mid (\mathbf{s})\mathbf{S}$$

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The barycentric boolean calculus

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Definition (λ + Barycentric + Boolean)

Atomic terms **s** and barycentric terms **T** are inductively defined

$$\mathbf{s} \quad ::= \quad \mathbf{x} \in \mathcal{V} \mid \lambda \mathbf{x}.\mathbf{s} \mid (\mathbf{s})\mathbf{S}$$

$$\mid \mathbf{T} : \mathcal{B} \mid \mathbf{F} : \mathcal{B} \mid \texttt{if } \mathbf{s} \texttt{ then } \mathbf{R} \texttt{ else } \mathbf{S} : \mathcal{B} \Rightarrow A \Rightarrow A \Rightarrow A$$

$$\mathbf{R}, \mathbf{S} \quad ::= \quad \sum_{i=1}^{m} a_i \, \mathbf{s}_i \qquad \text{where} \begin{cases} \forall i \leq m,\ a_i \in \Bbbk\,, \\ \sum_{i=1}^{m} a_i = 1\,. \end{cases}$$

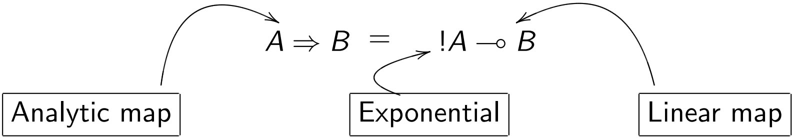
The booleans are affine combinations of `true` (**T**) and `false` (**F**) and $\mathcal{B} = 1 \oplus 1$ from linear logic.

Semantics of $\Lambda_{\mathcal{B}}$

- Webbed model enriched with coefficients and sums, hence **vector spaces**.
- Boolean type leads to **finite dimensional** vector spaces:

$$\llbracket \mathbf{T} \rrbracket = (1,0), \qquad \llbracket \mathbf{F} \rrbracket = (0,1),$$
$$\llbracket \texttt{if } (a\mathbf{T} + b\mathbf{F}) \texttt{ then } \mathbf{Q} \texttt{ else } \mathbf{R} \rrbracket = a \llbracket \mathbf{Q} \rrbracket + b \llbracket \mathbf{R} \rrbracket,$$
$$\llbracket \textstyle\sum a_i \, \mathbf{s}_i \rrbracket = \sum a_i \llbracket \mathbf{s}_i \rrbracket .$$

- Functional type leads to **infinite dimensional** vector spaces:



The web of exponential isn't finite, hence we need **topology**!

Non-determinism

For every proof π and π' of linear logic formulæ,

$$\frac{\pi \vdash A \quad \pi' \vdash A \multimap B}{[\pi\,;\,\pi'] \vdash B}\texttt{Cut} \qquad [\![\pi\,;\,\pi']\!] = \left(\sum_{a \in |A|} [\![\pi]\!]_a \, [\![\pi']\!]_{a,b}\right)_{b \in |B|}$$

The sum

- allows *non-determinism*, since result of different computations are added;
- is controlled since, in the simple typed case, it is finite.

Finiteness spaces use *orthogonality* between $\pi \vdash A$ and $\pi' \vdash A^{\perp}$

$$|[\![\pi]\!]| \cap |[\![\pi']\!]| \text{ finite.}$$

to make explicit the **controlled non-determinism**.

Relational Finiteness Spaces

Let $\mathcal{I}$ be countable, for each $\mathcal{F} \subseteq \mathcal{P}(\mathcal{I})$, let us denote

$$\mathcal{F}^{\perp} = \{u' \subseteq \mathcal{I} \; ; \; \forall u \in \mathcal{F},\; u \cap u' \text{ finite}\}\,.$$

Definition

A *relational finiteness space* is a pair $A = (|A|, \mathcal{F}(A))$ where the *web* $|A|$ is countable and the collection $\mathcal{F}(A)$ of finitary subsets satisfies $(\mathcal{F}(A))^{\perp\perp} = \mathcal{F}(A)$.

Example

Booleans.

$$\mathcal{B} = (\mathbb{B}, \mathcal{P}(\mathbb{B})) \text{ with } \begin{cases} \mathbb{B} & = \{\mathbf{T}, \mathbf{F}\} \\ \mathcal{P}(\mathbb{B}) & = \{\emptyset, \{\mathbf{T}\}, \{\mathbf{F}\}, \{\mathbf{T}, \mathbf{F}\}\} \end{cases}.$$

Integers.

$\mathcal{N} = (\mathbb{N}, \mathcal{P}_{fin}(\mathbb{N}))$ and $\mathcal{N}^{\perp} = (\mathbb{N}, \mathcal{P}(\mathbb{N}))$.

Linear Finiteness Spaces

Let $\Bbbk$ be an infinite discrete field. For every sequence $x \in \Bbbk^{|A|}$, the *support* of x is $|x| = \{a \in |A| \; ; \; x_a \neq 0\}$.

Definition

The *linear finiteness space* associated to $A = (|A|, \mathcal{F}(A))$ is

$$\Bbbk\langle A\rangle = \{x \in \Bbbk^{|A|} \; ; \; |x| \in \mathcal{F}(A)\}.$$

The *linearized topology* is generated by the neighborhoods of 0

$$V_J = \{x \in \Bbbk\langle A\rangle \; ; \; |x| \cap J = \emptyset\}, \quad \text{with } J \in \mathcal{F}(A)^{\perp}.$$

Example

Booleans. $\mathbb{B} = \{\mathbf{T}, \mathbf{F}\}$ $\quad \Bbbk\langle\mathcal{B}\rangle = \Bbbk^2$.
Integers. $|\mathcal{N}| = \mathbb{N}$ $\quad \Bbbk\langle\mathcal{N}\rangle = \Bbbk^{(\omega)}$ and $\Bbbk\langle\mathcal{N}^{\perp}\rangle = \Bbbk^{\omega}$.

Finiteness Spaces, Functions.

Theorem (Taylor-Ehrhard expansion)

Every program $P : A \Rightarrow B$ is interpreted by an analytic function $[\![P]\!] : \Bbbk\langle A\rangle \rightarrow \Bbbk\langle B\rangle$

$$P = \sum_{n\in\mathbb{N}} P^{(n)}(0)\ x^{\otimes n}.$$

Example

$$\begin{aligned}
\Bbbk\langle !\mathcal{B} \multimap 1\rangle &= \Bbbk\left[X_t, X_f\right], \\
\Bbbk\langle !\mathcal{B} \multimap \mathcal{B}\rangle &= \Bbbk\langle !\mathcal{B} \multimap 1 \oplus 1\rangle = \Bbbk\langle !\mathcal{B} \multimap 1\rangle^2 \\
&= \Bbbk\left[X_t, X_f\right] \times \Bbbk\left[X_t, X_f\right].
\end{aligned}$$

Towards completeness

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What is totality ?

A way to refine the semantics and step to full completeness.

For every proof π and π' of linear logic formulæ,

$$\frac{\pi \vdash A \qquad \pi' \vdash A^{\perp}}{[\pi \,;\, \pi'] \vdash \perp}\,\texttt{Cut} \qquad [\![\pi \,;\, \pi']\!] = \langle [\![\pi]\!] \,,\, [\![\pi']\!] \rangle = 1$$

Let A be a finiteness space $A = (|A|, \mathcal{F}(A))$. The associate linear space is $\mathbb{k}\langle A\rangle = \{x \in \mathbb{k}^{|A|} \;;\; |x| \in \mathcal{F}(A)\}$.

Definition

A totality candidate is an affine subspace $\mathcal{T}$ of $\mathbb{k}\langle A\rangle$ such that $\mathcal{T}^{\bullet\bullet} = \mathcal{T}$ with

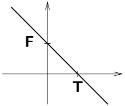
$$\mathcal{T}^{\bullet} = \{x' \in \mathbb{k}\langle A\rangle' \;;\; \forall x \in \mathcal{T},\, \langle x', x\rangle = 1\}\,.$$

A totality space is a pair $(A, \mathcal{T}(A))$ with $\mathcal{T}(A)^{\bullet\bullet} = \mathcal{T}(A)$.

An algebraic description

Every construction of linear logic has an algebraic description as a closed affine subspace.

$\mathcal{B} = 1 \oplus 1$ Affine combinations:



$f \in \mathcal{T}(A \multimap B)$ whenever $\forall x \in \mathcal{T}(A)$, $f(x) \in \mathcal{T}(B)$;

$F \in \mathcal{T}(A \Rightarrow B)$ whenever $\forall x \in \mathcal{T}(A)$, $F(x) \in \mathcal{T}(B)$.

Example ($\mathcal{B} \Rightarrow \mathcal{B} = !\mathcal{B} \multimap \mathcal{B}$)

$$\Bbbk\langle \mathcal{B} \Rightarrow \mathcal{B} \rangle = \Bbbk[X_t, X_f] \times \Bbbk[X_t, X_f],$$

$$(P_{\mathbf{T}}, P_{\mathbf{F}}) \in \mathcal{T}(\mathcal{B} \Rightarrow \mathcal{B}) \Leftrightarrow \forall (x_{\mathbf{T}}, x_{\mathbf{F}}) \in \mathcal{T}(\mathcal{B}),\ P_{\mathbf{T}}(x_{\mathbf{T}}, x_{\mathbf{F}}) + P_{\mathbf{F}}(x_{\mathbf{T}}, x_{\mathbf{F}}) = 1.$$

Full Completeness at the first order boolean type.

Syntax of $\mathbf{\Lambda}_{\mathcal{B}}$:

$$\begin{array}{rcl} \mathbf{s} & ::= & \mathbf{x} \in \mathcal{V} \mid \lambda \mathbf{x}.\mathbf{s} \mid (\mathbf{s})\mathbf{S} \\ & & \mid \mathbf{T} \mid \mathbf{F} \mid \texttt{if } \mathbf{s} \texttt{ then } \mathbf{R} \texttt{ else } \mathbf{S} \\ \\ \mathbf{R}, \mathbf{S} & ::= & \sum_{i=1}^{m} a_i \, \mathbf{s}_i \qquad \text{where } \sum_{i=1}^{m} a_i = 1\,. \end{array}$$

Proposition

For every term $\mathbf{S} \in \mathbf{\Lambda}_{\mathcal{B}}$ of type $\mathcal{B}^n \Rightarrow \mathcal{B}$, the semantics is total $[\![\mathbf{S}]\!] \in \mathcal{T}\langle \mathcal{B}^n \Rightarrow \mathcal{B} \rangle$.

Theorem (Completeness)

Every total function of $\mathcal{T}(\mathcal{B}^n \Rightarrow \mathcal{B})$ is the interpretation of a term of the boolean barycentric calculus.

Algebraic proof: Euclidean division and affine combinations.

Around

Corollary

Parallel functions such as

- Parallel-Or
- Berry function

can be encoded in $\Lambda_{\mathcal{B}}$.

Generalisations

- Possible for finitary types built over 1 and $\oplus$.
- Impossible for infinite types such as $\mathcal{N}$.
- Generalisation to higher order boolean type ?