Totality, towards completeness

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Contents

The barycentric boolean calculus

Definition

For all $n \in \mathcal{N}$, we define inductively the terms of $\mathbf{\Lambda}_{\mathcal{B}n}$ by

$$\mathbf{R}, \mathbf{S} ::= \sum_{i=1}^{m} a_i \ \mathbf{s}_i \quad \text{with} \sum_{i=1}^{m} a_i = 1, \text{ and}$$

$$\mathbf{s}, \mathbf{s}_i ::= \lambda \mathbf{x}_1 \ldots \mathbf{x}_n . \mathbf{x}_i \mid \mathbf{T} \mid \mathbf{F} \mid \texttt{if } \mathbf{s} \texttt{ then } \mathbf{S} \texttt{ else } \mathbf{R}.$$

Notice, that every barycentric boolean term is of type $\mathcal{B}^n \Rightarrow \mathcal{B}$

$$\sum a_i(\lambda \bar{\mathbf{x}} . \mathbf{s}_i) \simeq \lambda \bar{\mathbf{x}} . \sum a_i \mathbf{s}_i$$

$$\mathbf{T} \simeq \lambda \bar{\mathbf{x}} . \mathbf{T}, \qquad \mathbf{F} \simeq \lambda \bar{\mathbf{x}} . \mathbf{F},$$

$$\texttt{if } (\lambda \bar{\mathbf{x}} \, \mathbf{s}) \texttt{ then } (\lambda \bar{\mathbf{x}} \, \mathbf{S}) \texttt{ else } (\lambda \bar{\mathbf{x}} \, \mathbf{R}) \simeq \lambda \bar{\mathbf{x}} \texttt{ if } \mathbf{s} \texttt{ then } \mathbf{S} \texttt{ else } \mathbf{R}$$

Semantics of $\mathbf{\Lambda}_{\mathcal{B}n}$

Every $\mathbf{S} \in \mathbf{\Lambda}_{\mathcal{B}n}$, is interpreted by a pair

$$\left(\llbracket \mathbf{S} \rrbracket_t , \llbracket \mathbf{S} \rrbracket_f\right) \in \Bbbk\left[X_1, \dots, X_{2n}\right] \times \Bbbk\left[X_1, \dots, X_{2n}\right]$$

inductively defined by

$$\begin{aligned}
&\llbracket \textstyle\sum a_i\, \mathbf{s}_i \rrbracket = \sum a_i \llbracket \mathbf{s}_i \rrbracket ,\\
&\llbracket \mathbf{T} \rrbracket = (1, 0), \qquad \llbracket \mathbf{F} \rrbracket = (0, 1),\\
&\llbracket \lambda \mathbf{x}_1 \dots \mathbf{x}_n \cdot \mathbf{x}_i \rrbracket = (X_{2i-1}, X_{2i}),\\
&\llbracket \texttt{if } \mathbf{P} \texttt{ then } \mathbf{Q} \texttt{ else } \mathbf{R} \rrbracket = \Big(\llbracket \mathbf{P} \rrbracket_t \llbracket \mathbf{Q} \rrbracket_t + \llbracket \mathbf{P} \rrbracket_f \llbracket \mathbf{R} \rrbracket_t ,\\
&\qquad\qquad \llbracket \mathbf{P} \rrbracket_t \llbracket \mathbf{Q} \rrbracket_f + \llbracket \mathbf{P} \rrbracket_f \llbracket \mathbf{R} \rrbracket_f \Big).
\end{aligned}$$

Reduction

The reduction

$$\texttt{if } (a\,\mathbf{T} + b\,\mathbf{F}) \texttt{ then } \mathbf{R} \texttt{ else } \mathbf{S} \quad \to \quad a\,\mathbf{R} + b\,\mathbf{S}$$

Proposition (Soundeness)

Let $\mathbf{S} \in \mathbf{\Lambda}_{\mathcal{B}}$*. If* $\mathbf{S} \to \mathbf{T}$*, then* $[\![S]\!] = [\![T]\!]$.

Theorem (Computational adequacy)

Let $\mathbf{S} \in \mathbf{\Lambda}_{\mathcal{B}}$*. If* $[\![\mathbf{S}]\!] = (a, b)$*, then* $\mathbf{S} \to a\,\mathbf{T} + b\,\mathbf{F}$.

Boolean polynomials and completeness

Definition

Boolean polynomials are the pairs of polynomials (P, Q) such that there is $\mathbf{S} \in \mathbf{\Lambda}_{\mathcal{B}n}$ such that $[\![\mathbf{S}]\!] = (P, Q)$.

Boolean polynomials can be algebraically characterized.

Proposition

Let $\mathbf{S} \in \mathbf{\Lambda}_{\mathcal{B}n}$ *and* $(x_i) \in \mathbb{k}^{2n}$*.*

$$(\forall i,\ x_{2i-1} + x_{2i} = 1) \Rightarrow [\![\mathbf{S}]\!]_t\,(x_i) + [\![\mathbf{S}]\!]_f\,(x_i) = 1.$$

Reciprocally,

Theorem (Completeness)

For every $P, Q \in \mathbb{k}\,[X_1, \ldots, X_{2n}]$ *such that* $P + Q - 1$ *vanishes on the common zeros of* $X_{2i-1} + X_{2i} - 1$*, there is* $\mathbf{S} \in \mathbf{\Lambda}_{\mathcal{B}n}$ *with* $[\![\mathbf{S}]\!] = (P, Q)$*.*

Proof of completeness (1)

Some notations:

$$\begin{aligned} \neg\mathbf{S} &= \texttt{if}\ \mathbf{S}\ \texttt{then}\ \mathbf{F}\ \texttt{else}\ \mathbf{T}, \\ \mathbf{S}^{+} &= \texttt{if}\ \mathbf{S}\ \texttt{then}\ \mathbf{T}\ \texttt{else}\ \mathbf{T}, \\ \mathbf{S}^{-} &= \texttt{if}\ \mathbf{S}\ \texttt{then}\ \mathbf{F}\ \texttt{else}\ \mathbf{F}, \\ \mathbf{\Pi}_i &= \lambda \mathbf{x}_1, \dots, \mathbf{x}_n \cdot \mathbf{x}_i. \end{aligned}$$

Lemma (Basic pairs)

The pairs of polynomials (X_{2i}, X_{2i-1}), $(X_{2i-1} + X_{2i}, 0)$, $(1 - X_{2i-1}, X_{2i-1})$ *and* $(1 - X_{2i}, X_{2i})$ *are booleans.*

$$\begin{aligned} (X_{2-1}, X_{2i}) &= [\![\mathbf{\Pi}_i]\!], \\ (X_{2i}, X_{2i-1}) &= X_{2i-1} \cdot (1,0) + X_{2i} \cdot (0,1) \\ &= [\![\texttt{if}\ \mathbf{\Pi}_i\ \texttt{then}\ \mathbf{T}\ \texttt{else}\ \mathbf{F}]\!] \\ &= [\![\neg\mathbf{\Pi}_i]\!]. \end{aligned}$$

Proof of completeness (2)

Lemma (Affine pairs)

For every polynomial $P \in \Bbbk[X_1, \ldots, X_n]$, the pair of polynomials $(1-P, P)$ is boolean.

Let d be the degree of P.
If $d = 0$, then $(1-P, P) = (1-a, a) = [\![(1-a)\mathbf{T} + a\mathbf{F}]\!]$.
If $d > 0$ and $X^\mu = \prod X_i^{\mu_i}$ with $\mu_1 \geq 1$, then

$$\begin{aligned}(1-X^\mu, X^\mu) &= (1-X_1)\cdot(1,0) + \\ &\quad X_1 \cdot \left(1 - X_1^{\mu_1-1}\prod_{i\neq 1} X_i^{\mu_i}, X_1^{\mu_1-1}\prod_{i\neq 1} X_i^{\mu_i}\right) \\ &= [\![\texttt{if } \mathbf{\Xi}_1 \texttt{ then } \mathbf{T} \texttt{ else } \mathbf{\Xi}_{d-1}]\!] = [\![\mathbf{\Xi}_\mu]\!]\,.\end{aligned}$$

If $P = \sum a_\mu \prod X_i^{\mu_i}$, then

$$\begin{aligned}(1-P, P) &= \left(1-\sum a_\mu\right)(1,0) + \left(\sum a_\mu\right)(1-X^\mu, X^\mu) \\ &= [\![\left(1-\sum a_\mu\right)\mathbf{T} + \left(\sum a_\mu\right)\mathbf{\Xi}_\mu]\!]\,.\end{aligned}$$

Proof of completeness (3)

Lemma (Spanning polynomials)

Let $P \in \Bbbk[X_1, \dots, X_{2n}]$. If P vanishes on the zeros common to $X_{2i-1} + X_{2i} - 1$, then there are $Q_i \in \Bbbk[X_1, \dots, X_{2n}]$ such that $P = \sum_{i=1}^{n} Q_i(X_{2i-1} + X_{2i} - 1)$.

Change of variables

$$\left.\begin{array}{rl} Y_i & = X_{2i-1} + X_{2i} - 1 \\ Y_{i+n} & = X_{2i} \end{array}\right\} \Rightarrow P_Y(0, \dots, 0, y_{n+1}, \dots, y_{2n}) = 0.$$

Since $\Bbbk[Y_2, \dots, Y_{2n}][Y_1]$ is an euclidean ring, there are $Q \in \Bbbk[Y_1, \dots, Y_{2n}]$, $R \in \Bbbk[Y, \dots, Y_{2n}]$ such that

$$P_Y = Q_1 Y_1 +$$

$\forall (y_i) \in \Bbbk^n$, $R_n(y_{n+1}, \dots, y_{2n}) = 0$, hence if $\Bbbk$ is infinite

$$P_Y = \sum_{i=1}^{n} Q_i Y_i.$$

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Since $\Bbbk[Y_2, \dots, Y_{2n}][Y_1]$ is an euclidean ring, there are $Q_1 \in \Bbbk[Y_1, \dots, Y_{2n}]$, $R_1 \in \Bbbk[Y_2, \dots, Y_{2n}]$ such that

$$P_Y = Q_1 Y_1 + R_1$$

$\forall (y_i) \in \Bbbk^n$, $R_n(y_{n+1}, \dots, y_{2n}) = 0$, hence if $\Bbbk$ is infinite

$$P_Y = \sum_{i=1}^{n} Q_i Y_i.$$

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Since $\Bbbk[Y_2, \dots, Y_{2n}][Y_1]$ is an euclidean ring, there are $Q_i \in \Bbbk[Y_1, \dots, Y_{2n}]$, $R_2 \in \Bbbk[Y_{i+1}, \dots, Y_{2n}]$ such that

$$P_Y = Q_1 Y_1 + Q_2 Y_2 + R_2$$

$\forall (y_i) \in \Bbbk^n$, $R_n(y_{n+1}, \dots, y_{2n}) = 0$, hence if $\Bbbk$ is infinite

$$P_Y = \sum_{i=1}^{n} Q_i Y_i.$$

Proof of completeness (3)

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Let $P \in \Bbbk[X_1, \ldots, X_{2n}]$. If P vanishes on the zeros common to $X_{2i-1} + X_{2i} - 1$, then there are $Q_i \in \Bbbk[X_1, \ldots, X_{2n}]$ such that $P = \sum_{i=1}^{n} Q_i(X_{2i-1} + X_{2i} - 1)$.

Change of variables

$$\left.\begin{aligned} Y_i &= X_{2i-1} + X_{2i} - 1 \\ Y_{i+n} &= X_{2i} \end{aligned}\right\} \Rightarrow P_Y(0, \ldots, 0, y_{n+1}, \ldots, y_{2n}) = 0.$$

Since $\Bbbk[Y_2, \ldots, Y_{2n}][Y_1]$ is an euclidean ring, there are $Q_i \in \Bbbk[Y_1, \ldots, Y_{2n}]$, $R_n \in \Bbbk[Y_{n+1}, \ldots, Y_{2n}]$ such that

$$P_Y = Q_1 Y_1 + Q_2 Y_2 + \cdots + Q_n Y_n + R_n$$

$\forall (y_i) \in \Bbbk^n$, $R_n(y_{n+1}, \ldots, y_{2n}) = 0$, hence if $\Bbbk$ is infinite

$$P_Y = \sum_{i=1}^{n} Q_i Y_i.$$

Proof of completeness (the end)

Theorem (Completeness)

For every $P, Q \in \Bbbk[X_1, \ldots, X_{2n}]$ *such that* $P + Q - 1$ *vanishes on the common zeros of* $X_{2i-1} + X_{2i} - 1$, *there is* $t \in \mathbf{\Lambda}_{\mathcal{B}}$ *with* $[\![t]\!] = (P, Q)$.

Proof of completeness (the end)

Theorem (Completeness)

For every $P, Q \in \Bbbk[X_1, \ldots, X_{2n}]$ such that $P + Q - 1$ vanishes on the common zeros of $X_{2i-1} + X_{2i} - 1$, there is $t \in \mathbf{\Lambda}_{\mathcal{B}}$ with $[\![t]\!] = (P, Q)$.

Spanning: $P + Q - 1 = \sum_{i=1}^{n} Q_i(X_{2i-1} + X_{2i} - 1)$.

$$\begin{aligned}(P, Q) \quad = \quad & \textstyle\sum_{i=1}^{n} [(1 - Q_i) \cdot (1, 0) + Q_i \cdot (X_{2i-1} + X_{2i}, 0)] \\ & +(1 - Q, Q) - n(1, 0).\end{aligned}$$

Basic pairs: $[\![\mathbf{\Pi}_i^+]\!] = (X_{2i-1} + X_{2i}, 0)$,
Affine pairs: $[\![\mathbf{Q}]\!] = (1 - Q, Q)$.

$$\begin{aligned}(P, Q) \quad = \quad & [\![\textstyle\sum_{i=1}^{n} \left(\texttt{if } \mathbf{Q}_i \texttt{ then } \mathbf{T} \texttt{ else } \mathbf{\Pi}_i^+\right) \\ & +\mathbf{Q} - n\,\mathbf{T}]\!] \,,\end{aligned}$$

Where does it come from?

Thesis subject

To define a linear space model of linear logic.

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- Interest? Lots of intuitions of linear logic come from linear algebra.
- Difficulty? Because of exponential, infinite dimension appears, hence problem of reflexivity solved with topology.
- Other attempts?

[Blute96] *Linear Laüchli semantics*,

[Girard99] *Coherent Banach spaces*,

[Ehrhard02] *On Köthe sequence spaces and LL*,

[Ehrhard05] *Finiteness spaces*.

Where does it come from?

Thesis subject

To define a linear space model of linear logic.

- Interest? Lots of intuitions of linear logic come from linear algebra.
- Difficulty? Because of exponential, infinite dimension appears, hence problem of reflexivity solved with topology.
- My attempt: Linearly topologized spaces (Lefschetz),
  - a generalization of finiteness spaces,
  - a natural notion of totality.

  The boolean polynomials corresponds to the totality space associated to $!\mathcal{B} \multimap \mathcal{B}$.

Denotational semantics.

Linear Logic

$A, B :=$

$$\begin{array}{cccccc} & 0 & \mid & A \oplus B & \mid & A \& B \\ \mid & 1 & \mid & A \otimes B & \mid & A \parr B \\ \mid & A^\perp & \mid & !A & \mid & ?A. \end{array}$$

Reflexivity

$A^{\perp\perp} = A.$

Linear implication

$A \multimap B = A^\perp \parr B.$

Intuitionistic implication

$A \Rightarrow B = !A \multimap B.$

Finiteness space

A is interpreted by a linear space $\Bbbk\langle A\rangle$.

$\pi \vdash A$ is interpreted by a vector $[\![\pi]\!] \in \Bbbk\langle A\rangle$.

Totality space

A is interpreted by an affine subspace $\mathcal{T}(A)$ of $\Bbbk\langle A\rangle$.

$\pi \vdash A$ is interpreted by a vector $[\![\pi]\!] \in \mathcal{T}(A)$.

Relational Finiteness Spaces

Let $\mathcal{I}$ be countable, for each $\mathcal{F} \subseteq \mathcal{P}(\mathcal{I})$, let us denote

$$\mathcal{F}^{\perp} = \{u' \subseteq \mathcal{I} \,|\, \forall u \in \mathcal{F},\ u \cap u' \text{ finite}\}\,.$$

Definition

A *relational finiteness space* is a pair $A = (|A|, \mathcal{F}(A))$ where the *web* $|A|$ is countable and the collection $\mathcal{F}(A)$ of finitary subsets satisfies $(\mathcal{F}(A))^{\perp\perp} = \mathcal{F}(A)$.

Example

Booleans.

$$\mathcal{B} = (\mathbb{B}, \mathcal{P}(\mathbb{B})) \text{ with } \begin{cases} \mathbb{B} & = \{\mathbf{T}, \mathbf{F}\} \\ \mathcal{P}(\mathbb{B}) & = \{\emptyset, \{\mathbf{T}\}, \{\mathbf{F}\}, \{\mathbf{T}, \mathbf{F}\}\} \end{cases}.$$

Integers.

$$\mathcal{N} = (\mathbb{N}, \mathcal{P}_{fin}(\mathbb{N})) \text{ and } \mathcal{N}^{\perp} = (\mathbb{N}, \mathcal{P}(\mathbb{N})).$$

Linear Finiteness Spaces

For every $x \in \mathbb{k}^{|A|}$, the *support* of x is $|x| = \{a \in |A| \,|\, x_a \neq 0\}$.

Definition

The *linear finiteness space* associated to $A = (|A|, \mathcal{F}(A))$ is

$$\mathbb{k}\langle A\rangle = \{x \in \mathbb{k}^{|A|} \,|\, |x| \in \mathcal{F}(A)\}.$$

The *linearized topology* is generated by the neighborhoods of 0

$$V_J = \{x \in \mathbb{k}\langle A\rangle \,|\, |x| \cap J = \emptyset\}, \quad \text{with } J \in \mathcal{F}(A)^\perp.$$

Example

Booleans. $\mathbb{k}\langle \mathcal{B}\rangle = \mathbb{k}^2$.

Integers. $\mathbb{k}\langle \mathcal{N}\rangle = \mathbb{k}^{(\omega)}$ and $\mathbb{k}\langle \mathcal{N}^\perp\rangle = \mathbb{k}^{\omega}$.

Finiteness Spaces

A Linear Logic Model

$$A^{\perp} \quad \rightsquigarrow \quad \mathbb{k}\langle A\rangle'$$

$$0 \quad \rightsquigarrow \quad \{0\}$$

$$\left.\begin{array}{c} A \& B \\ A \oplus B \end{array}\right\} \quad \rightsquigarrow \quad \mathbb{k}\langle A\rangle \oplus \mathbb{k}\langle B\rangle$$

$$1 \quad \rightsquigarrow \quad \mathbb{k}$$

$$A \multimap B \quad \rightsquigarrow \quad \mathcal{L}_c(A, B)$$

$$A \otimes B \quad \rightsquigarrow \quad \mathbb{k}\langle A\rangle \otimes \mathbb{k}\langle B\rangle$$

$$!A \quad \rightsquigarrow \quad \mathbb{k}\langle !A\rangle$$

Finiteness Spaces

A Linear Logic Model

$$A^\perp \rightsquigarrow \Bbbk\langle A\rangle' \quad \Rightarrow \text{Reflexivity}$$

$$0 \rightsquigarrow \{0\}$$

$$\left.\begin{array}{c} A\& B \\ A\oplus B \end{array}\right\} \rightsquigarrow \Bbbk\langle A\rangle \oplus \Bbbk\langle B\rangle$$

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$$A\otimes B \rightsquigarrow \Bbbk\langle A\rangle \otimes \Bbbk\langle B\rangle$$

$$!A \rightsquigarrow \Bbbk\langle !A\rangle \quad \Rightarrow \text{Infinite dimension}$$

Exponentials

The relational finiteness space associated with $!A$ is

$$\begin{aligned} |!A| &= \mathcal{M}_{\mathrm{fin}}(|A|), \\ \mathcal{F}(!A) &= \left\{ M \subseteq \mathcal{M}_{\mathrm{fin}}(|A|) \mid \bigcup_{\mu \in M} |\mu| \in \mathcal{F}(A) \right\}. \end{aligned}$$

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Example

$$|\mathcal{B}| = \{\mathbf{T}, \mathbf{F}\} \qquad \mathcal{F}(\mathcal{B}) = \mathcal{P}(\{\mathbf{T}, \mathbf{F}\})$$

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Example

$$|\mathcal{B}| = \{\mathbf{T}, \mathbf{F}\} \qquad \mathcal{F}(\mathcal{B}) = \mathcal{P}(\{\mathbf{T}, \mathbf{F}\})$$

$$\begin{aligned} |?\mathcal{B}^{\perp}| &= |!\mathcal{B}| = \mathcal{M}_{\mathrm{fin}}(\mathbf{T}, \mathbf{F}) \simeq \mathbb{N}^2 \\ \mathcal{F}(!\mathcal{B}) &= \{ M \subseteq \mathcal{M}_{\mathrm{fin}}(\mathbf{T}, \mathbf{F}) \mid \cup_{\mu \in M} |\mu| \in \mathcal{F}(\mathcal{B}) \} = \mathcal{P}(\mathbb{N}^2) \\ \mathcal{F}(?\mathcal{B}^{\perp}) &= \left\{ M \subseteq \mathbb{N}^2 \mid \forall M' \subseteq \mathbb{N}^2,\ M \cap M' \text{ fin.} \right\} = \mathcal{P}_{fin}(\mathbb{N}^2) \end{aligned}$$

Exponentials

The linear finiteness space associated with $!A$ is

$$\Bbbk\langle !A\rangle = \left\{ z \in \Bbbk^{\mathcal{M}_{\mathrm{fin}}(|A|)} \,|\, \cup_{\mu \in |z|} |\mu| \in \mathcal{F}(A) \right\}.$$

Example

$$|\mathcal{B}| = \{\mathbf{T}, \mathbf{F}\} \qquad \mathcal{F}(\mathcal{B}) = \mathcal{P}(\{\mathbf{T}, \mathbf{F}\})$$

$$\begin{aligned} |?\mathcal{B}^{\perp}| &= |!\mathcal{B}| = \mathcal{M}_{\mathrm{fin}}(\mathbf{T}, \mathbf{F}) \simeq \mathbb{N}^2 \\ \mathcal{F}(!\mathcal{B}) &= \{ M \subseteq \mathcal{M}_{\mathrm{fin}}(\mathbf{T}, \mathbf{F}) \mid \cup_{\mu \in M} |\mu| \in \mathcal{F}(\mathcal{B}) \} = \mathcal{P}(\mathbb{N}^2) \\ \mathcal{F}(?\mathcal{B}^{\perp}) &= \left\{ M \subseteq \mathbb{N}^2 \mid \forall M' \subseteq \mathbb{N}^2,\ M \cap M' \text{ fin.} \right\} = \mathcal{P}_{fin}(\mathbb{N}^2) \end{aligned}$$

$$\begin{aligned} \Bbbk\langle !\mathcal{B}\rangle &= \left\{ z \in \Bbbk^{\mathbb{N}^2} \,|\, |z| \in \mathcal{P}(\mathbb{N}^2) \right\} &= \Bbbk\,(X_t, X_f) \\ \Bbbk\langle ?\mathcal{B}^{\perp}\rangle &= \left\{ z \in \Bbbk^{\mathbb{N}^2} \,|\, |z| \in \mathcal{P}_{fin}(\mathbb{N}^2) \right\} &= \Bbbk\,[X_t, X_f] \end{aligned}$$

Finiteness Spaces

Theorem (Taylor expansion)

For every $f \in \mathcal{L}_{\mathrm{c}}(\Bbbk\langle !A\rangle, \Bbbk\langle B\rangle)$, there is an analytic function ϕ such that $\forall x \in \Bbbk\langle A\rangle$, $\phi(x) \in \Bbbk\langle B\rangle$.

$$\forall b \in |B|,\ \phi_b(x) = \sum_{\mu} f_{\mu,b} x^{\mu} \qquad \text{with } x^{\mu} = \prod_{a} x_a^{\mu(a)}.$$

Example

$$\begin{aligned}
\Bbbk\langle !\mathcal{B} \multimap 1\rangle &= \Bbbk\langle ?\mathcal{B}^{\perp}\rangle = \Bbbk\left[X_t, X_f\right], \\
\Bbbk\langle !\mathcal{B} \multimap \mathcal{B}\rangle &= \Bbbk\langle !\mathcal{B} \multimap 1 \oplus 1\rangle = \Bbbk\langle !\mathcal{B} \multimap 1\rangle^2 \\
&= \Bbbk\left[X_t, X_f\right] \times \Bbbk\left[X_t, X_f\right].
\end{aligned}$$

What is totality ?

A way to refine the semantics of a calculus and a hope to have completeness.

Let A be a finiteness space $A = (|A|, \mathcal{F}(A))$.
The associate linear space is $\Bbbk\langle A\rangle = \{x \in \Bbbk^{|A|} \,|\, |x| \in \mathcal{F}(A)\}$.

Definition

A totality candidate is an affine subspace $\mathcal{T}$ of $\Bbbk\langle A\rangle$ such that $\mathcal{T}^{\bullet\bullet} = \mathcal{T}$ with

$$\mathcal{T}^{\bullet} = \{x' \in \Bbbk\langle A\rangle' \,|\, \forall x \in \mathcal{T},\ \langle x', x\rangle = 1\}\,.$$

A totality space is a pair $(A, \mathcal{T}(A))$ with $\mathcal{T}(A)^{\bullet\bullet} = \mathcal{T}(A)$.

A model of linear logic

A refinement of finiteness spaces.

Let $A \in$ LL and $\pi : A$ an affine linear logic proof.

$$[\![\pi]\!] \in \Bbbk\langle A\rangle.$$

We define by induction a totality candidate $\mathcal{T}(A)$ such that

$$[\![\pi]\!] \in \mathcal{T}(A).$$

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Some constructions

$$A^{\perp} \rightsquigarrow (\Bbbk\langle A\rangle' \,,\, \mathcal{T}(A)^{\bullet}),$$

with $\mathcal{T}(A)^{\bullet} = \{x' \in \Bbbk\langle A\rangle' \,|\, \forall x \in \mathcal{T}(A),\, \langle x', x\rangle = 1\}$.

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$$[\![\pi]\!] \in \mathcal{T}(A).$$

Some constructions

$$A \oplus B \rightsquigarrow (\Bbbk\langle A\rangle \oplus \Bbbk\langle B\rangle\,,\,\overline{\mathrm{aff}}(\mathcal{T}(A) \times \{0\} \cup \{0\} \times \mathcal{T}(B))).$$

Example

$$\begin{aligned} \mathcal{T}\langle\mathcal{B}\rangle &= \left\{(x_t, y_t) \in \Bbbk^2 \,|\, x_t + y_t = 1\right\}, \\ \mathcal{T}\langle\mathcal{B}^\perp\rangle &= \mathcal{T}\langle 1\&1\rangle = (1,1). \end{aligned}$$

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Let $A \in$ LL and $\pi : A$ an affine linear logic proof.

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$$[\![\pi]\!] \in \mathcal{T}(A).$$

Some constructions

$$A \multimap B \rightsquigarrow (\mathcal{L}_c(A, B)\,,\, \{f \mid f(\mathcal{T}(A)) \subseteq \mathcal{T}(B)\}).$$

Example

$$\begin{aligned}\mathcal{T}\langle \mathcal{B} \multimap \mathcal{B}\rangle &= \{f \in \mathcal{L}_c(\mathbb{k}^2, \mathbb{k}^2) \mid \\ &\qquad x_t + y_t = 1 \Rightarrow f(x_t, x_f) \in \mathcal{T}(\mathcal{B})\} \\ &= \{f_t, f_f \in \mathcal{L}(\mathbb{k}^2, \mathbb{k}) \mid \\ &\qquad x_t + y_t = 1 \Rightarrow f_t(x_t, x_f) + f_f(x_t, x_f) = 1\}\,.\end{aligned}$$

A model of linear logic

A refinement of finiteness spaces.
Let $A \in \mathrm{LL}$ and $\pi : A$ an affine linear logic proof.

$$[\![\pi]\!] \in \mathbb{k}\langle A\rangle.$$

We define by induction a totality candidate $\mathcal{T}(A)$ such that

$$[\![\pi]\!] \in \mathcal{T}(A).$$

Some constructions

$$!A \rightsquigarrow \left(\mathbb{k}\langle !A\rangle\,,\,\overline{\mathrm{aff}}\left\{x^{!} \mid x \in \mathcal{T}(A))\right\}\right.$$

Example

$$\begin{aligned}\mathcal{T}\langle !\mathcal{B}\rangle &= \overline{\mathrm{aff}}\left\{(x_t\,\mathbf{T} + y_f\,\mathbf{F})^{!} \mid x_t + y_f = 1\right\} \\ &= \overline{\mathrm{aff}}\left\{\sum_{p,q} x_t^p\, x_f^q \mid x_t + y_f = 1\right\}.\end{aligned}$$

Totality Spaces

Theorem (Taylor expansion)

For every $f \in \mathcal{T}\langle !A \multimap B\rangle$, the associated an analytic function $\phi : \Bbbk\langle A\rangle \Rightarrow \Bbbk\langle B\rangle$ satisfies

$$x \in \mathcal{T}\langle A\rangle \Rightarrow \phi(x) \in \mathcal{T}\langle B\rangle.$$

Example

$$\begin{aligned} \Bbbk\langle !\mathcal{B} \multimap \mathcal{B}\rangle &= \Bbbk\left[X_t, X_f\right] \times \Bbbk\left[X_t, X_f\right], \\ \mathcal{T}\langle !\mathcal{B} \multimap \mathcal{B}\rangle &= \{(P, Q) \in \Bbbk\left[X_t, X_f\right]^2 \mid \\ & \qquad x_t + y_t = 1 \Rightarrow P(x_t, y_t) + Q(x_t, y_t) = 1\}. \end{aligned}$$

Back to barycentric boolean lambda-calculus

Definition

We define inductively the terms of $\mathbf{\Lambda}_{\mathcal{B}}$ by

$$\mathbf{R}, \mathbf{S} ::= \sum_{i=1}^{m} a_i \, \mathbf{s}_i \quad \text{with} \sum_{i=1}^{m} a_i = 1, \text{ and}$$

$$\mathbf{s}, \mathbf{s}_i ::= \mathbf{x} \in \mathcal{V} \mid \lambda \mathbf{x}.\mathbf{s} \mid (\mathbf{s})\mathbf{S} \mid \mathbf{T} \mid \mathbf{F} \mid \texttt{if } \mathbf{s} \texttt{ then } \mathbf{S} \texttt{ else } \mathbf{R}.$$

Types

We consider only simply typed lambda-term with

$$\sum a_i \, \mathbf{s}_i^A : A, \qquad \mathbf{T}, \mathbf{F} : \mathcal{B},$$

$$\texttt{if } (-) \texttt{ then } (-) \texttt{ else } (-) : (\mathcal{B}^n \Rightarrow \mathcal{B})^3 \Rightarrow (\mathcal{B}^n \Rightarrow \mathcal{B}).$$

Notice that term of $\mathbf{\Lambda}_{\mathcal{B}^n}$ is a term of $\mathbf{\Lambda}_{\mathcal{B}}$ with type $\mathcal{B}^n \Rightarrow \mathcal{B}$.

Semantics

We use the translation of the *intuitionist implication* into linear logic

$$A \Rightarrow B \simeq !A \multimap B.$$

To each typed barycentric boolean term is associated a proof of affine linear logic.

$[\![\mathbf{S}]\!]$ is the semantics of the proof associated to $\mathbf{S}$.

Theorem

Let $\mathbf{S} \in \mathbf{\Lambda}_{\mathcal{B}}$. *If* $\mathbf{S}$ *of type* A, *then* $[\![\mathbf{S}]\!] \in \mathcal{T}\langle A \rangle$.

Soundness and partial completeness

Corollary

For every term $\mathbf{S} : \mathcal{B} \Rightarrow \mathcal{B} \simeq\, !\mathcal{B} \multimap \mathcal{B}$*,*
$[\![\mathbf{S}]\!] \in \mathcal{T}\langle !\mathcal{B} \multimap \mathcal{B}\rangle$ *which is equal to*
$\{(P,Q) \in \Bbbk\,[X_t, X_f]^2 \,|\, x_t + y_t = 1 \Rightarrow P(x_t,y_t) + Q(x_t,y_t) = 1\}$

Reciprocally, we have already seen

Theorem

For every pair of polynomials $(P,Q) \in \mathcal{T}\langle !\mathcal{B} \multimap \mathcal{B}\rangle$*, there is* $\mathbf{S} \in \mathbf{\Lambda}_{\mathcal{B}}$ *such that* $[\![\mathbf{S}]\!] = (P,Q)$*.*

This is a completeness theorem for first order boolean terms which has even been proved for $\otimes^n !\mathcal{B} \multimap \mathcal{B}$.

Conclusion

Completeness

- Total completeness for LL ?
  no, it is not even complete for MALL: $(\mathcal{B} \multimap \mathcal{B}) \multimap \mathcal{B}$
- Total completeness for higher order hierarchy $\mathbf{\Lambda}_{\mathcal{B}}$?
- How to complete $\mathbf{\Lambda}_{\mathcal{B}}$ to get completeness ?

Totality

Totality spaces constitute an elegant affine model of linear logic where linear logic construction are algebraically defined and completeness also seem to have an algebraic characterization.