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Mackey-complete spaces and power series – a topological model of differential linear logic

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In this paper, we describe a denotational model of Intuitionist Linear Logic which is also a differential category. Formulas are interpreted as Mackey-complete topological vector space and linear proofs are interpreted as bounded linear functions. So as to interpret non-linear proofs of Linear Logic, we use a notion of power series between Mackey-complete spaces, generalizing entire functions in $\mathbb{C}$. Finally, we get a quantitative model of Intuitionist Differential Linear Logic, with usual syntactic differentiation and where interpretations of proofs decompose as a Taylor expansion.

1. Introduction

Many denotational models of linear logic are discrete, based for example on graphs such as coherent spaces (Girard 1986), on games (Abramsky et al. 2000; Hyland and Ong 2000), on sets and relations, or on vector spaces with bases (Ehrhard 2002, 2005). This follows the intrinsic discrete nature of proof theory, and of linear logic. The computational interpretation of linearity in terms of resource consumption is still a discrete notion, proofs being seen as operators on multisets of formulas.

Besides, Ehrhard and Regnier show in Ehrhard and Regnier (2003b) and Ehrhard and Regnier (2003a) how it is possible to add a differentiation rule to Linear Logic, in this way constructing Differential Linear Logic (DiLL). In this work, differentiation is seen as a way of transforming a non-linear proof $f : A \Rightarrow B$ into a linear proof $Df : A \multimap (A \Rightarrow B)$. In models such as the relational model, differentiation has a combinatorial interpretation. In Ehrhard (2002) and Ehrhard (2005), non-linear proofs are interpreted as power series between Köthe spaces and Finiteness spaces respectively, that are sequence spaces. However, differentiation is historically of a continuous nature. In continuous models of DiLL, where non-linear proofs are interpreted as differentiable functions, the syntactic differentiation corresponds to the mathematical one. It is a fairly natural question to look for a continuous semantics of linear logic in which the differential rule can be interpreted.

Bornologies. The search for topological models of Linear Logic relies on some fundamental mathematical issues. Indeed, having a cartesian closed category of topological spaces is not straightforward. Several answers exists (see Escardó and Heckmann (2001/02) for a past account), and amongst them is the definition of convenient spaces and smooth functions by Frölicher, Kriegl, and Michor in Frölicher and Kriegl (1988) and Kriegl and Michor (1997). Those are the smooth functions used in Blute et al. (2012) for modelling DiLL. Moreover, as explained by Girard in the introduction of Girard (1999), if the proofs are interpreted by continuous functions, then, notably, the interpretations of the proofs of $A, A \Rightarrow B \vdash B$ and of $A \vdash (A \Rightarrow B) \Rightarrow B$ are also continuous. That is, $x, f \mapsto f(x)$ and $x \mapsto (\delta_x : f \mapsto f(x))$ must be continuous. This would be the case if linear function spaces bore both a uniform convergence and a pointwise convergence topology. We believe that this is solved by the use of bounded sets, i.e. by using the advantages of the theory of bornologies (see Hogbe-Nlend (1977) for an overview of this theory). Indeed, the Banach–Steinhauss theorem says that between Banach spaces, the topology of uniform convergence on bounded sets and the pointwise convergence topology on a space of linear functions give rise to the same bounded sets. This theorem is generalized in Kriegl and Michor (1997), where the authors use Mackey-complete spaces (complete spaces for a specific version of Cauchy sequences) and bounded linear maps (linear maps preserving bounded sets). This observation was exploited in Frölicher and Kriegl (1988) and Kriegl and Michor (1997) where bounded linear functions replace continuous ones.

Quantitative semantics. Introduced by Girard in Girard (1988), quantitative semantics refine the analogy between linear functions and linear programs (consuming exactly once their input). Indeed, programs consuming exactly n-times their resources are seen as monomials of degree n. General programs are described as the disjunction of their executions consuming n-times their resources. Mathematically, this means that non-linear programs are interpreted by potentially infinite sums of monomials, that are power series. This analogy can be found in many denotational models of variant of Linear Logic such as Fock spaces (Blute et al. 1994), Köthe spaces (Ehrhard 2002), Finiteness spaces (Ehrhard 2005), Probabilistic Coherent spaces (Danos and Ehrhard 2016), or, in a more categorical setting, in analytic functors (Hasegawa 2002) and generalized species (Fiore et al. 2008).

This line of research has also given rise to models characterizing quantitative properties of non-deterministic, probabilistic, or resource sensitive PCF extensions. For instance, in Ehrhard et al. (2014), probabilistic coherent spaces are shown to be fully abstract for probabilistic PCF. In Laird et al. (2013b), weighted relational models are used to compare programs with respects to how many different ways or with which probability they compute a result. In Pagani et al. (2014), the quantitative framework is successfully used to design a model of higher order quantum computation. In Laird et al. (2013a), the authors define differential cartesian closed categories based on categories of games.

Mackey-complete spaces and Power series. In this paper, we have brought to light a model of Intuitionist DiLL, whose objects are *locally convex topological vector spaces* that are Mackey-complete (see Definition 2.5). The ingredients of the model have been chosen so that they correspond cleanly to the constructions of DiLL: For instance, vector spaces are used to interpret linearity, and topology to interpret differentiation. This is a quantitative version of the work on Mackey-complete spaces and smooth maps by Blute et al. (2012).

We use the notion of bounded set when we ask *linear functions* not to be continuous but *bounded* , that is to send a bounded set to a bounded set. The two notions are closely related, but distinct. As a consequence, the interpretation of the *negation* is based on the bounded dual and not on the usual continuous dual.

The *multiplicative conjunction* $\otimes$ of Linear Logic is interpreted by the *bounded* tensor product of topological vector spaces which has to be Mackey-completed.

The *additive conjunction* & and *disjunction* $\oplus$ are interpreted respectively by the cartesian product and the coproduct in the category of Mackey-complete spaces and bounded linear functions. Finite products and coproducts coincide, so that the category is equipped with finite biproducts. Notice that if we wanted to ensure that the bounded dual of infinite products are coproducts (the reverse comes automatically), we would need to work with spaces whose cardinals are not strongly inaccessible (Jarchow 1981, 13.5.4). This assumption is not restrictive as it is always possible to construct a model of ZFC with non-accessible cardinals.

Non-linear proofs of DiLL are interpreted as *power series*, that are sums of bounded n-monomials. In order to work with these functions, we must make use of the theory of holomorphic maps developed in the second chapter of Kriegl and Michor (1997). This is made possible since the spaces we consider are in particular Mackey-complete. We have proven that the category of Mackey-complete spaces and power series is cartesian closed, by generalizing the Fubini theorem over distributions $S(E \times F, \mathbb{C}) \simeq S(E, S(F, \mathbb{C}))$ and by using interchange of converging summations in $\mathbb{C}$. The exponential modality is interpreted as a Mackey-complete subspace of the bounded dual of the space of scalar power series. Indeed, any space can be embedded in its bounded bidual $!E \subset (!E)^{\times\times} = (!E \multimap \perp)^{\times}$ and using the key decomposition $!E \multimap \perp \simeq E \Rightarrow \perp = S(E, \mathbb{C})$ of Linear Logic gives us that $!E \subseteq S(E, \mathbb{C})^{\times}$. Finally, because we are working with topological vector spaces, the interpretation of the co-dereliction rule of DiLL is the operator taking the directed derivative at 0 of a function.

Related works. Our model follows a long history of models establishing connections between analyticity and computability.

Fock spaces (Blute et al. 1994) and Coherent Banach spaces (Girard 1999) were the first step towards a continuous semantics of Linear Logic. More precisely, Fock spaces are Banach spaces and Coherent Banach spaces are dual pairs of Banach spaces (see Jarchow 1981, Chap. 8 for an overview of the theory of dual pairs). In Fock spaces, linear programs are interpreted as contractive bounded linear maps and general programs as holomorphic or analytic functions. Similarly, in Coherent Banach spaces, linear programs are interpreted as continuous linear functions and general programs as bounded analytic functions defined on the open unit ball. Yet, neither Fock spaces nor Coherent Banach spaces are completely a model of the entire linear logic, but they are a model of a linear exponential, that is of weakening. However, both works were already using bounded sets (e.g. bounded linear forms and continuous linear forms correspond on Banach spaces) and we take advantage of replacing Banach spaces norms with bornologies.

With Köthe spaces (Ehrhard 2002) and then Finiteness spaces (Ehrhard 2005), Ehrhard designed two continuous semantics of Linear Logic. The objects of the two models are sequence spaces equipped with a structure of topological vector spaces. Köthe spaces are locally convex spaces over the usual real or complex fields, whereas Finiteness spaces are endowed with a linearized topology over a field (potentially of reals or complexes) endowed with discrete topology. The linear proofs are interpreted as continuous linear functions and the non-linear ones as analytic mappings. Notice that the interpretation of a linear logic formula enjoys an intrinsic characterization, even if these models are related to the relational semantics. Indeed, a Linear Logic formula is interpreted as a space of sequences whose indices constitute its relational interpretation. Furthermore, the interpretation of a proof is a sequence whose support (the indices of non-zero coefficients) is its relational interpretation. Although interpretations of formulas may differ, proofs are identically interpreted in Köthe or Finiteness models (and in the model presented in the present paper). The main difference between our model and these Köthe or Finiteness spaces models is precisely that Mackey-complete spaces do not have to be sequence spaces. They digress from the discrete setting of the relational model. Since Köthe spaces and Mackey-complete spaces are both endowed with locally convex topology, one could think that the first are a special case of the last. However, the function spaces are endowed with the compact open topology for Köthe spaces and with the bounded open topology for the Mackey-complete spaces. In particular, the dual $E_X^\perp$ of a Köthe Space E_X is isomorphic to the topological dual of E_X, which is in general a strict subset of the bornological dual (all Köthe spaces are not bornological). It raises an interesting question about whether a description with bounded subsets would help having an intrinsic description of Köthe spaces. On the contrary, although Finiteness spaces do not have the same kind of topology, their use of bounded sets is central and our model borrows a lot of Finiteness spaces constructions.

The present work is thought as a version of Convenient spaces (Blute et al. 2012), that is Mackey-complete spaces and smooth maps, without the bornological condition on the topology. In this model of Intuitionist Linear Logic, which is a differential category, non-linear proofs are interpreted with some specific smooth maps. No references are made to a discrete setting. As in Finiteness spaces, the topology and the bornology are dually related. Although this bornological condition facilitates the proofs, it is not necessary to interpret Intuitionnist Linear Logic. Thus, in our model, we release the bornological condition on the topology.

In many Quantitative models of Linear Logic, as in Normal functors (Girard 1988; Hasegawa 2002), Fock spaces (Blute et al. 1994), or Finiteness spaces (Ehrhard 2005, 2007), non-linear proofs are interpreted as analytic functions. In our model, we refine smooth maps into analytic ones. On the way, we consider topological vector spaces over $\mathbb{C}$ to be able to handle holomorphic functions. This is another difference with Convenient Vector spaces as presented in Blute et al. (2012).

Content of the paper. We begin the paper by laying down the bornological setting (Subsection 2.1) and by defining the central notion of Mackey-complete spaces (Subsection 2.2). Then, in Section 3, we begin the definition of the model by the linear category of Mackey-complete spaces and bounded linear maps that is cartesian and symmetric monoidal closed. This linear part is the base of the present work, but also of the model of Mackey-complete spaces and smooth functions introduced in Blute et al. (2012). We have given an overview of this work in a slightly different setting in Section 4 in order to properly describe the landscape of our work. Finally, in Section 5, we introduce the power series, their definition and properties that are useful in demonstrating that Mackey-complete spaces and power series constitute a quantitative model of Intuitionistic DiLL.

2. Preliminaries

2.1. *Topologies and bornologies*

Let us first set the topological scene. We will handle *complex* topological vector spaces. We denote by $\mathbb{C}$ the field of complex numbers and by $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. (When working with linear maps, one could describe a monoidal structure either with complex or real vector spaces. However, in Section 5, we study power series and make an extensive use of their holomorphic properties.)

More precisely, we will work with *locally convex separated topological vector spaces* (see Jarchow (1981, I.2.1)) and refer to them as *lctvs*. From now on, E and F denote lctvs. A set C in a $\mathbb{C}$-vector space is said to be *absolutely convex* when for all $x, y \in C$, for all $\lambda, \mu \in \mathbb{C}$, if $|\lambda| + |\mu| < 1$, then $\lambda x + \mu y \in C$. By definition, the topology of an lctvs is generated by a basis of neighbourhood of 0 made of absolutely convex subsets. We will use that if C is an absolutely convex subset of an lctvs, then $\bar{C} \subset 3C$, and $\lambda C + \mu C \subset (\lambda + \mu)C$ for all $\lambda, \mu \in \mathbb{C}$.

Bounded sets. We will also work with *bornologies*, that is collections of bounded sets with specific closure properties. A *subset* b of an lctvs is *bounded* when it is absorbed by every 0-neighbourhood U, that is there is $\lambda \in \mathbb{C}$ such that $b \subseteq \lambda U$. A *disk* is a bounded absolutely convex set. A *function* is *bounded* when it sends a bounded set of its domain to a bounded set of its codomain. Two spaces are *bounded equivalent*, noted $E \simeq F$, when there is a bijection $\phi : E \to F$ such that ϕ and ϕ^{-1} are both linear and bounded.

Let us denote E' the space of linear *continuous* forms on E, $E^\times$ the space of linear *bounded* forms on E, and $E^\star$ the space of linear forms on E. Remark that any linear continuous function is bounded and so $E' \subset E^\times \subset E^\star$.

The Mackey–Arens Theorem. It is a fundamental theorem for the theory of bornologies. It states that bounded subsets can be characterized as the ones that are sent to a bounded ball by any continuous linear form. We state it for bounded linear forms.

Lemma 2.1. A subset $b \subset E$ is bounded if and only if it is scalarly bounded, that is

$$\forall \ell \in E^\times,\ \exists \lambda > 0,\ |\ell(b)| < \lambda.$$

Proof. By definition of the bounded linear forms, the image of a bounded set is bounded. For the reverse implication, we use the Mackey–Arens theorem (see e.g. Schaefer (1971, IV.3.2)). Indeed, since for any $\ell \in E'$, $\ell \in E^\times$, we have $\ell(b)$ is bounded, and so b is bounded. □

The Hahn–Banach Theorem. Usually, the Hahn–Banach *separation* theorem is stated for continuous linear forms (see Jarchow 1981, Proposition 7.2.2.a). We adapt it to bounded linear forms as $E' \subseteq E^\times$. The principal flaw to the theory of vectorial spaces and bornologies is that there is no version of the Hahn–Banach *extension* theorem for bounded linear maps (Hogbe-Nlend 1970).

Proposition 2.2. Let C be a closed convex subset of E. If $x \in E \setminus C$, then there is $\ell \in E' \subset E^\times$ such that $|\ell(x)| = 1$ and for all $y \in C$, $|\ell(y)| = 0$.

As a corollary that we will frequently use, let $x \in E$ and $b \subset E$, if for all $\ell \in E^\times$, $\ell(x) \in \ell(b)$, then $x \in b$.

Bornivorous subsets. We introduced bounded sets as a definition depending on the topology. It is also possible to define 0-neighbourhood from the notion of bounded set.

Definition 2.3. A **bornivorous** set is a subset $U \subseteq E$ absorbing any bounded subset up to dilatation: $\forall b \subset E$ bounded, $\exists \lambda \in \mathbb{R}^+$, $\lambda b \subseteq U$. The **bornological topology** τ_b of E is the topology generated by the bornivorous disks of E.

Note that any neighbourhood of 0 in the topology of E is bornivorous, but the converse is false, i.e. the bornivorous topology τ_b is finer than the topology of E. The point of the bornologification of an lctvs is precisely to enrich E with all the bornivourous subsets as 0-neighbourhood, so that we get better relations between continuity and boundedness (see Jarchow 1981, 13.1 for details on this notion).

Proposition 2.4.

1. The bounded subsets of E and of $\tau_b(E)$ are the same.
2. A linear function $f : E \to F$ is bounded if and only if $f : \tau_b(E) \to F$ is continuous.

Proof. The first item stems from definition handling. For the second one, if $f : \tau_b(E) \to F$ is continuous, it is bounded and because E and $\tau_b(E)$ bears the same bounded sets $f : E \to F$ is bounded. Conversely, suppose that $f : E \to F$ is bounded. Then, one can see that when V is a 0-neighbourhood in F, $f^{-1}(V)$ is a bornivorous subset of E; hence, a 0-neighbourhood in $\tau_b(E)$. Thus, $f : \tau_b(E) \to F$ is continuous. □

2.2. *Mackey-complete spaces*

Mackey-complete spaces are very common spaces in mathematics as Mackey-completeness is a very weak completeness condition. For example, every complete space, quasi-complete, or weakly complete space is Mackey-complete. Mackey-complete spaces are called locally complete spaces in Jarchow (1981), or convenient spaces in Kriegl and Michor (1997). Although it is not a very restraining notion, Mackey-completeness suffices to speak about smoothness between lctvs, in the meaning of Kriegl and Michor.

Definition 2.5. Consider E an lctvs. A *Mackey-Cauchy net* in E is a net $(x_\gamma)_{\gamma\in\Gamma}$ such that there is a net of scalars $\lambda_{\gamma,\gamma'}$ decreasing towards 0 and a bounded set b of E such that

$$\forall \gamma, \gamma' \in \Gamma, x_\gamma - x_{\gamma'} \in \lambda_{\gamma,\gamma'} b.$$

A space where every Mackey–Cauchy net converges is called **Mackey-complete**.

Note that a converging Mackey–Cauchy net does in fact *Mackey-converge*, i.e. there is a net of scalars λ_γ decreasing towards 0 such that $x_\gamma - \lim_\gamma x_\gamma \in \lambda_\gamma b$. Note also that a *Mackey-converging* net is always a *converging* net, by definition of boundedness in an lctvs.

Notice that the convergence of Mackey–Cauchy nets and the convergence of Mackey–Cauchy sequences are equivalent (see Kriegl and Michor 1997, I.2.2). Mackey-converging sequences and bounded functions behave particularly well together. Indeed, a bounded function is not continuous in general, so it does not preserve converging sequences but it does preserve Mackey–Cauchy nets.

Proposition 2.6. Bounded linear functions preserve Mackey-convergence and Macke–Cauchy nets.

There is a nice characterization of Mackey-completeness, through a decomposition into a collection of Banach spaces.

Definition 2.7. Consider b an absolutely convex and bounded subset of an lctvs E. We write E_b for the linear span of b in E, and it is a normed space, when endowed with the Minkowski functional defined as $p_b(x) = \inf\{\lambda \in \mathbb{R}^+ \mid x \in \lambda b\}$.

As a Mackey–Cauchy net is nothing but a Cauchy net in some specific E_b, we have:

Proposition 2.8 (Kriegl and Michor (1997, I.2.2)). An lctvs E is Mackey-complete if and only if for every closed bounded and absolutely convex subset b, E_b is a Banach space.

Similarly to what happens in the more classical theory of complete spaces, we have a *Mackey-completion* procedure. This one is slightly more intricate than the completion procedure, as it consists of the right completion of each of the E_b.

Proposition 2.9 (Kriegl and Michor (1997, I.4.29)). For every lctvs E, there is a Mackey-complete lctvs $\tilde{E}$ and a bounded embedding $\imath : E \to \tilde{E}$, unique up to bounded isomorphism, such that for every Mackey-complete lctvs F, for every bounded linear map $f : E \to F$, there is a unique bounded linear map $\tilde{f} : \tilde{E} \to F$ extending f such that $f = \tilde{f} \circ \imath$.

Beware that the *Mackey-closure* procedure does not behave as simply as the closure procedure. Indeed, the Mackey-closure of a subset B is the smallest Mackey-closed (i.e. closed for Mackey-convergence) set containing X. It does not coincide in general with the Mackey-adherence of X, that is the set of all limits of Mackey-converging sequences of elements of X, see Kriegl and Michor (1997, I.4.32).

Let us describe finally a few preservation properties of Mackey-complete spaces.

Proposition 2.10 (Kriegl and Michor (1997, I.2.15)). Mackey-completeness is preserved by limits, direct sums, strict inductive limits of sequences of closed embeddings. It is not preserved in general by quotient nor general inductive limits.

Spaces of bounded maps. Let us write $\mathcal{B}(E, F)$ for the space of bounded maps from E to F (not necessarily linear), endowed with the topology of uniform convergence on bounded sets of E. As in the linear case (see below), bounded sets of $\mathcal{B}(E, F)$ are the *equibounded* ones, that is the sets $B \subset \mathcal{B}(E, F)$ such that for any $b \subset E$ bounded, $B(b) = \{f(x) \mid f \in B,\ x \in b\}$ is bounded in F.

Proposition 2.11 (Kriegl and Michor (1997, I.2.15)). Let E and F be lctvs. If F is Mackey-complete, then so is $\mathcal{B}(E, F)$.

3. A symmetric monoidal closed and cartesian category

Let us write **Lin** for the category whose objects are Mackey-complete spaces, and whose morphisms are linear bounded maps. In this setting, the additives are interpreted using product and coproduct, whilst the multiplicative connectives are interpreted using a tensor product and its dual. The only tricky point is to find a good tensor product in our category: This is possible thanks to the Mackey-completion procedure.

3.1. *The (co)cartesian structure*

Topological products and coproducts.

The **cartesian product** of a countable family of Mackey-complete spaces is Mackey-complete when endowed with the product topology (Kriegl and Michor 1997, I.2.15). A subset of the cartesian product is bounded if and only if it is bounded in each direction. The terminal object $\top$ is the one-point vector set $\{0\}$ viewed as a Mackey-complete space.

The **coproduct** of a countable family of Mackey-complete spaces is Mackey-complete when endowed with the coproduct topology, that is the finest topology on $\bigoplus_i E_i$ for which the injections $E_i \to \bigoplus_i E_i$ are continuous. Then, $B \subset \bigoplus_i E_i$ is bounded if and only if $\{i \mid \exists x \in B \cap E_i\}$ is finite and if for every i, $B \cap E_i$ is bounded in E_i. The $\{0\}$ vector space is also the unit 0 of the coproduct.

Notice that in the finite case, the product and the coproduct coincide algebraically and topologically. In the infinite case, the distinction between product and coproduct corresponds to the distinction between the space of complex sequences $\mathbb{C}^{\mathbb{N}} = \prod_{n \in \mathbb{N}} \mathbb{C}$ and the space of complex finite sequences $\mathbb{C}^{(\mathbb{N})} = \bigoplus_{n \in \mathbb{N}} \mathbb{C}$. In $\mathbb{C}^{\mathbb{N}}$ bounded sets are the ones included in an arbitrary product of disks, whereas in $\mathbb{C}^{(\mathbb{N})}$ bounded sets are included in a finite product of disks.

Duality. The bounded isomorphism $(\oplus_{i \in I} E_i)^\times = \prod_{i \in I} E_i^\times$ always holds. Indeed, the restriction to each E_i of a morphism $f \in (\oplus_{i \in I} E_i)^\times$ gives a family $(f_i) \in \prod_{i \in I} E_i^\times$. Conversely, any family $(f_i) \in \prod_{i \in I} E_i^\times$ transforms into a sum $\sum_i f_i \in (\oplus_{i \in I} E_i)^\times$ which is pointwise convergent as it is applied to finite sequences of terms. The dual isomorphism $(\prod_{i \in I} E_i)^\times = \oplus_{i \in I} E_i^\times$ holds only in certain cases.

Proposition 3.1. If I is countable, then $(\prod_{i \in \mathbb{N}} E_i)^\times = \oplus_{i \in \mathbb{N}} E_i^\times$.

Proof. Let us first consider $h \in \oplus_{i \in \mathbb{N}} E_i^\times$, we can define $h_i \in E_i^\times$ the ith components of h, so that $h = \sum_{i \in \mathbb{N}} h_i$. As a finite sum, $h \in (\prod_{i \in \mathbb{N}} E_i)^\times$.

Now, consider $f \in (\prod_{i \in \mathbb{N}} E_i)^\times$ and let us write $f_i : E_i \to \mathbb{C}$ for $f_{|\{0\} \times \ldots \{0\} \times E_i \times \{0\} \times \ldots}$, that is the restriction of f to E_i. f_i is bounded. Let us show that there is only a finite number of

i such that f_i is not null. Indeed, if this is not the case, there is a non-decreasing sequence $(i_k) \in \mathbb{N}^{\mathbb{N}}$ and for any $k \in \mathbb{N}$, $x_k \in E_{i_k}$ such that $f(0, \dots, 0, x_k, 0, \dots) = f_{i_k}(x_k) > k$. Remark that the set $\{(0, \dots, 0, x_k, 0, \dots) \mid k \in \mathbb{N}\}$ is bounded in $\prod_{i\in\mathbb{N}} E_i$, since f is bounded, we get a contradiction.

Let $h = \sum_{i\in\mathbb{N}} f_i$. We have just proved that $h \in \oplus_{i\in\mathbb{N}} E_i^{\times}$, so that h is bounded as a finite sum of bounded functions. Notice that $h = \sum_{i\in\mathbb{N}} f_i \in (\prod_{i\in\mathbb{N}} E_i)^{\times}$. Let us now show that $g = f - h$ is null. Remark that for any $i \in \mathbb{N}$, the restriction of g to E_i is null. Suppose that $g \neq 0$. There is $x \in \prod_{i\in\mathbb{N}} E_i$ such that $g(x) = 0$. Consider i maximal such that there is $x \in \{0\} \times \dots \{0\} \times \prod_{k\geqslant i} E_k$ such that $g(x) = 0$. Then, $g(x) = g_i(x_i) + g_{|\prod_{k>i} E_k}((x_k)_{k>i})$. As $g_i(x_i) = 0$, we have $g_{|\prod_{k>i} E_k}((x_k)_{k>i}) = 0$, and thus $g(0, \dots, 0, x_{i+1}, x_{i+2}, \dots) = 0$. This contradicts the maximality of i. Then, $g = 0$, and $f = h \in \oplus_{i\in\mathbb{N}} E_i^{\times}$. □

There is a generalization of this proposition. Thanks to the Mackey–Ulam theorem (Robertson 1970; Ulam 1930), when the cardinal I indexing the family is not strongly inaccessible, then the bounded dual of the product is the coproduct of the bounded duals.

3.2. *The monoidal structure*

The *bounded tensor product* (Kriegl and Michor 1997, I.5.7) $E \otimes_\beta F$ is the algebraic tensor product with the finest locally convex topology such that $E \times F \to E \otimes F$ is bounded. The **complete bounded tensor product** $E \hat{\otimes} F$ is the Mackey-completion of $E \otimes_\beta F$. The tensor product is associative. The bounded sets associated with this topology are generated by $b_E \otimes b_F$ for b_E and b_F, respectively bounded in E and F. The unit 1 is the base field $\mathbb{C}$ endowed with its usual topology.

The space of **linear bounded functions** $\mathcal{L}(E, F)$ is endowed with the *bounded open* topology, generated by $\mathcal{W}(b, V) = \{f \in \mathcal{L}(E, F) \mid f(b) \subset V\}$, where b is bounded in E and V is open in F. The associated bornology is generated by the *equibounded* sets, that is the $B \subset \mathcal{L}(E, F)$ such that for any bounded b in E, $B(b)$ is bounded in F. Indeed, consider $B \subset \mathcal{L}(E, F)$ bounded for the topology of uniform convergence on bounded set. Consider $b \subset E$ a bounded set and $V \subset F$ a 0-neighbourhood in F. As B is bounded, there is $\lambda \in \mathbb{C}$ such that $B \subset \lambda\mathcal{W}(b, V)$, that is $B(b) \subset \lambda V$. Thus, $B(b)$ is bounded in F. Conversely, consider $B \subset \mathcal{L}(E, F)$ an equibounded set, b a bounded in E and $V \subset F$ a 0-neighbourhood in F. Then, there is $\lambda \in \mathbb{C}$ such that $B(b) \subset \lambda V$, that is $B \subset \lambda\mathcal{W}(b, V)$. Thus, B is bounded in $\mathcal{L}(E, F)$.

Proposition 3.2. Let E and F be lctvs. If F is Mackey-complete, then so is $\mathcal{L}(E, F)$.

Proof. Thanks to Proposition 2.11, a Mackey–Cauchy net in $\mathcal{L}(E, F)$ converges into a bounded map from E to F. Besides, the limit of a net of linear functions is also linear. □

Let E, F, G be locally convex spaces. Endowed with the bounded open topology, the space of **bounded bilinear mappings**, denoted as $\mathcal{L}(E, F; G)$, is also locally convex.

Proposition 3.3. The bornological tensor product is the solution of the universal problem of linearizing bounded bilinear mappings. More precisely, for any $h \in \mathcal{L}(E, F; G)$, there is a unique $h_\beta \in \mathcal{L}(E \otimes_\beta F, G)$ such that

$$\begin{tikzcd} E \times F \arrow[r] \arrow[d, "h"] & E \otimes_\beta F \arrow[dl, dashed, "h_\beta"] \\ G & \end{tikzcd}$$

Proof. Consider E, F, G, and h as in the proposition. Let us define $h_\beta : x \otimes y \mapsto h(x, y)$. We see that h_β is linear and bounded. The uniqueness of h_β follows from the universal property of $E \otimes F$ in the category of vector spaces and linear map. □

If moreover, G is Mackey-complete, then so is $\mathcal{L}(E, F; G)$ (for the same reason as in the proof of Proposition 3.2). Then, the universal property diagram can be extended through the Mackey-completion universal property, for any $h \in \mathcal{L}(E, F; G)$, there is a unique $\hat{h} \in \mathcal{L}(E \hat{\otimes} F, G)$ such that

$$\begin{tikzcd} E \times F \arrow[r] \arrow[d, "h"] & E \otimes_\beta F \arrow[r] \arrow[dl, dashed, "h_\beta"'] & E \hat{\otimes} F \arrow[dll, dashed, "\hat{h}"] \\ G & & \end{tikzcd}$$

Proposition 3.4. $E \otimes_\beta -$ is left adjoint to $\mathcal{L}(E, -)$, i.e. for any locally convex spaces E, F, and G, there are natural isomorphisms:

$$\mathbf{Lin}(E, \mathcal{L}(F, G)) \simeq \mathcal{L}(E, F; G) \simeq \mathbf{Lin}(E \otimes_\beta F, G).$$

This property extends to the complete case by the universal property of the Mackey-completion. If E, F, and G are Mackey-complete, then

$$\mathbf{Lin}(E, \mathcal{L}(F, G)) \simeq \mathcal{L}(E, F; G) \simeq \mathbf{Lin}(E \hat{\otimes} F, G).$$

Proof. (See Kriegl and Michor 1997, I.5.7) The bijection $\mathbf{Lin}(E \otimes_\beta F, G) \simeq \mathcal{L}(E, F; G)$ follows from the universal property of the bornological tensor product. Besides, both the bijection and its inverse are bounded. The canonical bijections between $\mathbf{Lin}(E, \mathcal{L}(F, G))$ and $\mathcal{L}(E, F; G)$ are bounded and natural in every elements E, F, and G. □

The next theorem follows from the symmetry and the associativity of the tensor product, and from Propositions 3.2 and 3.4:

Theorem 3.5. The category **Lin** of Mackey-complete spaces and linear bounded maps endowed with the complete bounded tensor product $\hat{\otimes}$ is symmetric monoidal closed.

4. Smooth maps in topological vector spaces

Mackey-complete spaces have already been at the heart of a model of the differential extension of the Intuitionist Linear Logic (Blute et al. 2012), inspired by the work presented in Frölicher and Kriegl (1988), Kriegl and Michor (1997). In this model, spaces are interpreted as Mackey-complete *bornological* spaces, i.e. spaces such that topologies and bornologies are mutually induced. Non-linear proofs are interpreted as smooth maps.

Actually, the bornological condition of Frölicher and Kriegl (1988), Blute et al. (2012) can be removed as in Kriegl and Michor (1997). In particular, the characterization of open sets as bornivorous sets is not necessary. Nevertheless, constructions such as tensor product or exponential use Mackey-completion that give rise to bornological spaces.

Moreover, as underlined in Kriegl and Michor (1997, II.7.1), any complex locally convex space can be seen as a real convex space endowed with a linear complex structure $J : E \to E$ defined as $J(x) = i\,x$ and the complex scalar multiplication is then given by $(\lambda + i\,\mu)\,x = \lambda\,x + \mu\,J(x)$. The only adaptation consists in replacing absolutely convex sets by $\mathbb{C}$-absolutely convex ones. Moreover, a $\mathbb{C}$-linear functional l is characterized by its real part $\mathrm{Re}\circ l$, since $l(x) = (\mathrm{Re}\circ l)(x) + i(\mathrm{Re}\circ l)(J(x))$. Thus, considerations on smooth curves as well as concepts used in Kriegl and Michor (1997), Blute et al. (2012) still hold in the complex setting.

We now present an overview of Blute et al. (2012) in the complex setting and where the bornological condition has been relaxed. This settle the general framework and category in which our model is built in the next Section.

4.1. *Smooth curves and smooth maps*

Let E be a Mackey-complete space. As in any topological space and for any curve $c : \mathbb{R} \to E$, the derivative can be defined as usual:

$$c'(t) = \lim_{s\to 0} \frac{c(t+s) - c(t)}{s}.$$

Then, such a **curve** is **smooth** whenever it is infinitely derivable. Let us write $\mathcal{C}_E$ for the set of smooth curves into E. It is endowed with the topology of uniform convergence on bounded sets of each derivative separately. A basis of 0-neighbourhood for this topology is made of $\mathcal{W}_{b,i,U}$, where b is a bounded set in $\mathbb{R}$, $i \in \mathbb{N}$, U is a 0-neighbourhood in E, and

$$\mathcal{W}_{b,i,U} = \{c \mid \forall t \in b,\ c^{(i)}(t) \in U\}.$$

Proposition 4.1 (Kriegl and Michor (1997, I.3.7)). *E is Mackey-complete if and only if $\mathcal{C}_E$ is Mackey-complete.*

Proof. By considering the set of all its derivative, one can see $\mathcal{C}_E$ as a closed subspace of $\prod_n \mathcal{B}(\mathbb{R}, E)$. Conversely, E can be identified as the closed subspace of $\mathcal{C}_E$ given by the constant curves. □

A set of curves $C \subset \mathcal{C}_E$ is *bounded* whenever each derivative is uniformly bounded on bounded subsets of $\mathbb{R}$ (see Kriegl and Michor 1997, I.3.9):

$$\forall i, \forall b \subset \mathbb{R} \text{ bounded},\ \exists b_E \text{ bounded in } E,\ \text{such that } \{c^{(i)}(x) \mid c \in C,\ x \in b\} \subset b_E.$$

Let $\mathcal{C}^\infty(E, F)$ denote the space of **smooth maps** from E to F, i.e. $f : E \to F$ preserving smooth curves: $\forall c \in \mathcal{C}_E,\ f \circ c \in \mathcal{C}_F$. This definition of smoothness is a generalization of the usual one for finite dimension topological vector spaces (see Boman 1967).

Proposition 4.2 (Kriegl and Michor (1997, I.3.11)). When F is Mackey-complete, then $\mathcal{C}^\infty(E,F)$ is also Mackey-complete.

Proof. The space $\mathcal{C}^\infty(E,F)$ can be seen as the closed subspace of $\prod_{c\in\mathcal{C}_E}\mathcal{C}_F$ whose elements $(f_c)_c$ are those such that for every $g\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, $f_{c\circ g}=f_c\circ g$. □

A subset B of $\mathcal{C}^\infty(E,F)$ is *bounded* whenever, the image $c^*(B)=\{f\circ c \mid c\in B\}$ of any curve $c\in\mathcal{C}_E$, is bounded in $\mathcal{C}_F$.

There is a strong link between boundedness and smoothness. First, smoothness only depend on the bounded subsets (see Kriegl and Michor 1997, I.1.8). So that, if two different topologies on E induce the same bounded subsets, then the set of smooth curves into E are identical. Moreover, the space of bounded linear maps can be embedded in the space of smooth ones.

Proposition 4.3 (Kriegl and Michor (1997, I.2.11)). The linear bounded maps between E and F are exactly the smooth linear ones.

It is not the case for continuity and boundedness, since a bounded linear map has not to be continuous. Indeed, consider an infinite-dimensional Banach space B, and the same space endowed with its weak topology B_w. By Lemma 2.1, the identity function $\mathrm{id}: B_w\to B$ is bounded. But as the weak topology is strictly coarser than the norm topology, id is not continuous. Though any continuous linear map is bounded and so smooth.

Notice that the bounded open topology of $\mathcal{L}(E,F)$ coincides with the topology induced by $\mathcal{C}^\infty(E,F)$ (see Kriegl and Michor 1997, I.5.3), so that $\mathcal{L}(E,F)$ can be seen as a closed linear subspace of $\mathcal{C}^\infty(E,F)$ (see Kriegl and Michor 1997, I.3.17).

4.2. *A model of differential linear logic*

One of the great interest of smooth maps as defined above is that they lead to a cartesian closed category (Kriegl and Michor 1997, I.3.12). Let **Smooth** denote the cartesian closed category of *Mackey-complete spaces* and *smooth maps*. In Blute et al. (2012), it is shown that there is a linear–non-linear adjunction between **Lin** and **Smooth**, so defining a model of Intuitionistic Linear Logic.

An adjunction between Lin and Smooth. The exponential in Blute et al. (2012) is generated by evaluation maps on certain smooth functions, whilst in our case, it is generated by evaluation maps on certain power series (see Definition 5.31).

Let us introduce the *Dirac delta distribution* δ. For any Mackey-complete space E and $x\in E$, δ is the bounded and linear map (see Blute et al. 2012, Lemma 5.1) defined as

$$\delta:\begin{cases} E\to\mathcal{C}^\infty(E,\mathbb{C})^\times\\ x\mapsto\delta_x: f\mapsto f(x)\end{cases}$$

Actually, δ is smooth. In Section 5.2, we will construct a similar function δ from E to the dual of a space of power series, that is a power series.

The Dirac delta distributions are linearly independent (see Blute et al. 2012, Lemma 5.3). Hence, they form a basis of the linear span of the set $\delta(E)=\{\delta_x \mid x\in E\}$. The Mackey-complete space $!E$ is the Mackey-closure of this linear subspace of $\mathcal{C}^\infty(E,\mathbb{C})^\times$.

$$\frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ (\text{promotion}) \qquad \begin{array}{c} \dfrac{\vdash \Gamma}{\vdash \Gamma, ?A}\ (\text{weakening}) \\ \dfrac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ (\text{contraction}) \\ \dfrac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ (\text{dereliction}) \end{array} \qquad \begin{array}{c} \dfrac{\vdash \Gamma}{\vdash \Gamma, !A}\ (\text{co-weakening}) \\ \dfrac{\vdash \Gamma, !A, !A}{\vdash \Gamma, !A}\ (\text{co-contraction}) \\ \dfrac{\vdash \Gamma, A}{\vdash \Gamma, !A}\ (\text{co-dereliction}) \end{array}$$

Fig. 1. Exponential groups of LL (promotion, weakening, contraction, dereliction) and DiLL ((co-)weakening, (co-)contraction, (co-)dereliction)).

Let $f \in \mathbf{Lin}(E, F)$ be a smooth map. Its exponential $!f \in \mathbf{Lin}(!E, !F)$ is defined on the set $\delta(E)$ by $!f(\delta_x) = \delta_{f(x)}$. It is then extended to the linear span of $\delta(E)$ by linearity and to $!E$ by the universal property of the Mackey-completion.

The exponential functor ! enjoys a structure of comonad, which is defined on the Dirac delta distributions and then extended: The counit ϵ is the natural transformation given by the linear map $\epsilon_E \in \mathbf{Lin}(!E, E)$, defined as $\epsilon(\delta_x) = x$, the comultiplication ρ has components $\rho_E \in \mathbf{Lin}(!E, !!E)$ given as $\rho_E(\delta_x) = \delta_{\delta_x}$.

Theorem 4.4 (Blute et al. (2012)). The cokleisli category of the comonad ! over **Lin** is the category **Smooth**. In particular, for any Mackey-complete spaces E and F,

$$\mathbf{Lin}(!E, F) \simeq \mathbf{Smooth}(E, F).$$

Theorem 5.40 is similar but for the category **Series** of Mackey-complete spaces and power series.

A differential category. Working with smooth functions allows the author of Blute et al. (2012) to handle a notion of differentiation, which coincides with the usual notion. This makes **Lin**, endowed with !, a differential category (Blute et al. 2006), and a model of **Intuitionistic Differential Linear Logic** (Ehrhard 2016) (DiLL). Indeed, DiLL differs from Linear Logic by a more symmetric exponential group, where the usual promotion rule is replaced by three new rules: co-weakening, co-dereliction, and co-contraction (see Figure 1). Differential categories, and their co-kleisli counterpart, the cartesian differential categories (Blute et al. 2009), are thought of as axiomatizing the structure necessary to perform differential calculus. Models of DiLL are basically differential categories whose exponential is endowed with a bialgebraic structure.

In **Smooth**, finite products coincide with finite coproducts. This biproduct structure is transported by the strong monoidal functor ! to a bialgebra structure: $\Delta : !E \to !E \hat{\otimes} !E$ which is defined on Dirac distributions as $\Delta(\delta_x) = \delta_x \otimes \delta_x$, $e : !E \to \mathbb{C}$ is defined as $e(\delta_x) = 1$, $\nabla : !E \hat{\otimes} !E \to !E$ is given as $\nabla(\delta_x \otimes \delta_y) = \delta_{x+y}$ and $m^0 : \mathbb{C} \to !E$ is defined as $m^0(1) = \delta_0$.

Differentiation is constructed from the bialgebra structure and from a more primitive differentiation operator, denoted as coder $\in \mathbf{Lin}(E, !E)$. This operator is the interpretation of the co-dereliction and corresponds to the differentiation at 0 of a smooth map:

$$\text{coder}(v) = \lim_{t \to 0} \frac{\delta_{tv} - \delta_0}{t}.$$

The differential operator is then interpreted as the usual one in analysis:

$$d : \mathcal{C}^\infty(E,F) \to \mathcal{C}^\infty(E,\mathbf{Lin}(E,F)) \qquad df(x)(v) = \lim_{t\to 0} \frac{f(x+tv)-f(x)}{t}.$$

In Section 5.5, we will use similar morphisms to show that **Series**, made of Mackey-complete spaces and power series, is a differential category. Moreover, we will interpret co-dereliction as the operator extracting the first monomial from a power series.

5. A quantitative model of linear logic

The purpose of this paper is to define a new quantitative model of DiLL, with a strong analytical flavour. Indeed, one of the characteristic of the quantitative models (Danos and Ehrhard 2016; Ehrhard 2002, 2005; Girard 1988; Hasegawa 2002) is that the morphisms in the cokleisli enjoy a Taylor expansion. The authors of Blute et al. (2012) constructed a smooth interpretation of DiLL, that we would like to refine into a quantitative model. We could have used a study of holomorphic and real analytic maps by Kriegl and Michor (1997, Chapter II): The construction of a model of holomorphic or real analytic maps should be easily done by following the constructions of Blute et al. (2012). However, these maps corresponds only locally to their Taylor development. As the interpretation of locality in denotational semantics remains unclear, we want to interpret the non-linear proofs of DiLL as functions corresponding in every point with their Taylor development at 0.

We take advantage of the fact that our spaces are Mackey-complete so as to define a quite general notion of power series which are in particular smooth (see Proposition 5.27). A power series is a converging sum of monomials. Indeed, a power series in $\mathbb{C}$ is represented by a sum $\sum_n a_n x^n$ converging pointwise on some disk. We are going to use power series between topological vector spaces; thus, the description has to be a little bit more involved and a power series will be a sum $\sum_n f_n$, where f_n is n-homogeneous and $\sum_n f_n(x)$ converges for every $x \in E$. Moreover, we need a stronger notion than pointwise convergence, so as to compose power series and to get a cartesian closed category. This is the uniform convergence on bounded sets of the partial sums $\sum_{n=0}^N f_n$, which will allow us to deeply relate weak, strong, and pointwise convergence of power series (see Proposition 5.21). As the space of power series between Mackey-complete spaces is Mackey-complete (see Proposition 5.28), we obtain a cartesian closed category of Mackey-complete spaces and power series between them.

To get to this point, we use a description of power series as functions sending holomorphic maps to holomorphic maps, and for this, proofs of Kriegl and Michor (1997) are adapted. This study gives us a Cauchy inequality on power series, and equivalences between weak convergence and strong convergence of power series, inspired from Bochnak and Siciak (1971). Finally, using weak convergence, we obtain the cartesian closedeness of the category.

The part on holomorphic maps between lctvs is not needed at first reading, as it is only used in the proof of Proposition 5.19. The reader may then skip Section 5.2.

5.1. *Monomials and power series*

Definition 5.1 (Kriegl and Michor (1997, I.5.15)). A function $f_n : E \to F$ is an n-**monomial** when there is $\tilde{f}_n$ an n-linear bounded function from E^n to F such that

$$\forall x \in E,\ f_n(x) = \tilde{f}_n \underbrace{(x, \dots, x)}_{n \text{ times}}.$$

We write $\mathcal{L}^n(E, F)$ for the space of n-monomials from E to F, and $\mathcal{L}(E^{\otimes^n}, F)$ for the space of bounded n-linear maps from E to F. We endow $\mathcal{L}(E^{\otimes^n}, F)$ with the locally convex topology of uniform convergence on bounded sets of E. As in the linear case (see Section 3.2), bounded sets of $\mathcal{L}^n(E, F)$ are the equibounded ones.

The following polarization formula relates the values of a monomial with the values of the unique multilinear map it comes from.

Lemma 5.2. Consider $f_n \in \mathcal{L}^n(E, F)$, and consider $\tilde{f}_n$ an n-linear map such that $\tilde{f}_n(x, \dots, x) = f_n(x)$. Then, for every $x_1, \dots, x_n \in E$:

$$\tilde{f}_n(x_1, \dots, x_n) = \tfrac{1}{n!} \sum_{\epsilon_1, \dots, \epsilon_n = 0}^{1} (-1)^{n - \sum_{j=1}^{n} \epsilon_j} f_n \left(\sum_{j=1}^{n} \epsilon_j x_j \right).$$

Proof. The proof relies on the expansion of the right-hand side by multilinearity and symmetry of $\tilde{f}_n$ (see Kriegl and Michor 1997, II.7.13). □

As in the case of bounded linear functions (see Proposition 2.6), monomials behave particularly well with respect to Mackey-convergence.

Lemma 5.3. Consider $(x_\gamma)_{\gamma \in \Gamma}$ a Mackey-converging net in E and $f_k : E \to F$ a k-monomial. Then, $f_k(x_\gamma)$ is a Mackey-converging net; thus, a converging net.

Proof. Let $\tilde{f}_k$ be the symmetric bounded k-linear map corresponding to f_k. Let $b \subset E$ be a bounded set, $x \in E$ and $(\lambda_{\gamma \in \Gamma}) \in \mathbb{C}^{\mathbb{N}}$ be a sequence decreasing towards 0 such that

$$\forall \gamma,\ x_\gamma - x \in \lambda_\gamma b.$$

Let us write $b' = \tilde{f}_k(b \times \cdots \times b)$. Then, for every $\gamma \in \Gamma$, we can factorize $f_k(x_\gamma) - f_k(x)$ following the classical equality $x^k - y^k = (x - y)(x^{k-1} + x^{k-2} y + \cdots + y^{k-1})$. Indeed,

$$f_k(x_\gamma) - f_k(x) = \tilde{f}_k(x_\gamma - x, x_\gamma, \dots, x_\gamma) + \tilde{f}_k(x_\gamma - x, \dots, x_\gamma, \dots, x_\gamma, x) + \tilde{f}_k(x_\gamma - x, x, \dots, x).$$

As $\tilde{f}_k$ is bounded, and as for every $\gamma \in \Gamma$, x_γ belongs to the bounded set $Mb + \{x\}$ for some M, there is a bounded b' in F such that

$$\forall \gamma,\ f_k(x_\gamma) - f_k(x) \in \lambda_\gamma b'.$$

□

An n-**homogeneous** function is a map f such that $f(\lambda x) = \lambda^n f(x)$ for any scalar λ.

Lemma 5.4 (Kriegl and Michor (1997, I.5.16.1)). A function f from E to F is an n-monomial if and only if it is a smooth n-homogeneous map.

Proof. As bounded n-linear functions are smooth by Proposition 4.3, n-monomials are smooth n-homogeneous functions. Conversely, by deriving at 0 an n-homogeneous smooth function along the curve $t \mapsto tx$, we can show that it is equal to its nth-derivative which is n-linear. □

Proposition 5.5. If F is Mackey-complete, then so is $\mathcal{L}^n(E,F)$.

Proof. There is a bounded isomorphism between the space $\mathcal{L}^n(E,F)$ and the space of all n-linear symmetrical morphisms from E to F, when the last one is endowed with the topology of uniform convergence on bounded sets of $E \times \cdots \times E$. Indeed, one associate an n-monomial to an n-linear symmectric morphism by applying it n-times to the same argument. Thanks to the Polarization Formula (see Lemma 5.2), we can obtain an n-linear symmetric morphism $\tilde{f}_n$ from an n-monomial f_n.

The mappings $(f_n \mapsto \tilde{f}_n)$ and $(\tilde{f}_n \mapsto f_n)$ preserves uniformly bounded sets; thus, $\mathcal{L}^n(E,F)$ and the space of all n-linear symmetrical morphisms from E to F are isomorphic. By definition of the symmetric nth-tensor product $E^{\otimes_s^n}$, the space $\mathcal{L}^n(E,F)$ is also isomorphic to $\mathcal{L}(E^{\otimes_s^n},F)$. This space is Mackey-complete as F is (see Proposition 3.2), and thus $\mathcal{L}^n(E,F)$ is also Mackey-complete.

□

Definition 5.6. A function f from E to F is a **power series** when f is *pointwise* equal to a converging sum of k-monomials:

$$\forall x,\ f(x) = \sum_{k=0}^{\infty} f_k(x),$$

and when this sum converges *uniformly on bounded* sets of E.

We write $S(E,F)$ for the space of **power series** between E and F and endow it with the topology of uniform convergence on bounded subsets of E.

Proposition 5.7. A power series is bounded.

Proof. Consider $f = \sum_k f_k \in S(E,F)$, b a bounded set in E, and U an absolutely convex 0-neighbourhood in F. We know that $\sum_k f_k$ converges uniformly on b. Hence, there is an integer N such that $(f - \sum_{k=1}^N f_k)(b) \subset U$. Besides, each f_k sends b to a bounded set, thus $(\sum_{k=1}^N f_k)(b)$ is bounded as a finite sum of bounded sets. So there is $\lambda \in \mathbb{C}$ such that $(\sum_{k=1}^N f_k)(b) \subset \lambda U$. Finally, $f(b) \subset (\lambda + 1)U$. □

5.2. *Power series and holomorphy*

We are going to show that if $f = \sum_n f_n : E \to F$ is a power series converging uniformly on bounded sets, it is holomorphic, according to the specific definition of Kriegl and Michor (1997, II.7.19). This definition is a generalization of the well-known definition of holomorphy for complex functions of a complex variable, and leads to a Cauchy inequality for f (see Proposition 5.19). This Cauchy inequality will turnout to be essential in showing cartesian closedeness and the composition results in Section 5.3.

This formula will result in the Mackey-convergence of power series (see Proposition 5.20), and will allow us to compose bounded linear forms with power series (see Proposition 5.25).

From now on, we are going to work with linear *continuous* forms $\ell \in E'$ in order to be able to write $\ell \circ (\sum_n f_n) = \sum_n \ell \circ f_n$.

Holomorphic curves. This part on holomorphic curves in an lctvs is inspired by the first theorem of Grothendieck (1953) and by Part 7 of Kriegl and Michor (1997) on Mackey-complete spaces and holomorphic functions.

Remember that a holomorphic curve in $\mathbb{C}$, $c : \mathbb{C} \to \mathbb{C}$ is a complex everywhere derivable function. It is infinitely many times differentiable, and verifies the Cauchy formula and the Cauchy inequality. For any $z' \in \mathbb{C}$ and any sufficiently small r,

$$\frac{c^{(n)}(z')}{n!} = \frac{1}{2\pi i} \int\limits_{|z-z'|=r} \frac{c(z)}{(z-z')^{n+1}} dz \qquad \text{and} \qquad \left|\frac{c^{(n)}(z')}{n!}\right| \leqslant \left|\frac{sup\{c(z) \mid |z-z'|=r\}}{r^n}\right|.$$

Moreover, it can be uniquely decomposed as a power series:

$$\forall a \in \mathbb{C}, \forall z \in \mathbb{C},\ c(z+a) = \sum_n \frac{c^{(n)}(a)}{n!} z^n.$$

We now give two approaches to holomorphic curves, that we then show equivalent.

Definition 5.8. A **strong holomorphic** curve $c : \mathbb{C} \to E$ is an everywhere complex derivable function. A **weak holomorphic** curve $c : \mathbb{C} \to E$ is a function such that for every $\ell \in E'$, $\ell \circ c$ is holomorphic.

Lemma 5.9. Let $c : \mathbb{C} \to E$ be a curve.

1. If c is strong holomorphic, then
$$\forall \ell \in E', \ell \circ c \text{ is complex derivable and } \forall z \in \mathbb{C},\ (\ell \circ c)'(z) = \ell(c'(z)).$$
2. If c is weak holomorphic, then c is bounded.
3. If c is weak holomorphic, then for all $z \in \mathbb{C}$, the difference quotient $(\frac{c(z+h)-c(z)}{h})_{h\in\mathbb{D}}$ is a Mackey–Cauchy net.

Proof. Let c be a strong holomorphic curve.

1. Let $\ell \in E'$. Since ℓ is linear and continuous, we have

$$\lim_{h\to 0} \frac{l \circ c(z+h) - \ell \circ c(z)}{h} = \ell(c'(z)).$$

Then, $\ell \circ c$ is complex derivable and $\forall z \in \mathbb{C}$, $(\ell \circ c)'(z) = \ell(c'(z))$.

Now, let c be a weak holomorphic curve.

2. Let b be a bounded set in $\mathbb{C}$ and $\bar{b}$ its closed absolutely convex closure. For every $\ell \in E'$, $(\ell \circ c)(\bar{b})$ is compact as the image in $\mathbb{C}$ of a compact set by a continuous function ($\ell \circ c$ is complex holomorphic and thus continuous). Then, $c(b)$ is weakly bounded and so bounded by Proposition 2.1.

3. This proof is adapted from Kriegl and Michor (1997, I.2.1). By translating c, we may assume that $z = 0$. For any $\ell \in E'$, $\ell \circ c$ is holomorphic in $\mathbb{C}$; hence, infinitely complex-derivable and $\ell \circ c$ is Lipschitz continuous. Then, we have

$$\begin{aligned}\frac{1}{z_1 - z_2}\left(\frac{l \circ c(z_1) - \ell \circ c(0)}{z_1} - \frac{l \circ c(z_2) - \ell \circ c(0)}{z_2}\right) &= \int_0^1 \frac{(l \circ c)'(rz_1) - (l \circ c)'(rz_2)}{z_1 - z_2} dr \\ &= \int_0^1 \frac{(l \circ c)'(rz_1) - (l \circ c)'(rz_2)}{rz_1 - rz_2} rdr\end{aligned}$$

Moreover, the curve $r \mapsto \frac{(\ell \circ c)'(rz_1) - (\ell \circ c)'(rz_2)}{rz_1 - rz_2}$ is locally bounded as $(l \circ c)$ is holomorphic. The set $\left\{\frac{1}{z_1 - z_2}\left(\frac{c(z_1) - c(0)}{z_1} - \frac{c(z_2) - c(0)}{z_2}\right) \mid z_1, z_2 \in \mathbb{D}\right\}$ is then scalarly bounded and thus bounded by Proposition 2.1. This is equivalent to show that the difference quotient is Mackey–Cauchy (see Definition 2.5). □

Proposition 5.10. The strong holomorphic curves into a Mackey-complete space are exactly the weak holomorphic curves.

Proof. A strong holomorphic curve is weak holomorphic by Property 1 of Lemma 5.9.

Now, let $c : \mathbb{C} \to E$ be a weak holomorphic curve into a Mackey-complete space E. Then, by the third property of the preceding lemma, for all $z \in \mathbb{C}$, the difference quotient $(\frac{c(z+h) - c(z)}{h})_{h \in \mathbb{D}}$ is Mackey–Cauchy and thus converges in E, since it is Mackey-complete. Hence, c is complex derivable and its derivative $c'(z)$ is the limit of the difference quotient. □

From now on, a **holomorphic curve** is either a weak or strong holomorphic curve.

Lemma 5.11. Let b be an absolutely convex and closed subset of E, γ be a path in $\mathbb{C}$ and $f : \mathbb{C} \to E$ be continuous. If for any $z \in \gamma([0;1])$, $f(z) \in b$, then the integral of f on the path γ is in b.

Proof. As $\int_\gamma f = \int_0^1 f(\gamma(t))dt$, this integral can be computed as the limit of the Riemann sums over $[0;1]$ of $f \circ \gamma$. As b is absolutely convex, each of these sums is in b. As it is closed, we have also $\int_\gamma f \in b$. □

Proposition 5.12. Let $c : \mathbb{C} \to E$ be a holomorphic curve. There is an absolutely convex, closed bounded subset b of E such that, if $\mathbb{D}$ denotes the closed unit ball in $\mathbb{C}$:

$$c(\mathbb{D}) \subset b \text{ and } \forall n \in \mathbb{N},\ c^{(n)}(\mathbb{D}) \subset n!b.$$

Proof. Thanks to Property 1 of Lemma 5.9, c is bounded. This justifies the existence of b such that $c(\mathbb{D}) \subset b$. Moreover, for every $\ell \in E'$, the curve $\ell \circ c$ is holomorphic in $\mathbb{C}$ according to Proposition 5.10. Thus, for every $z \in \mathbb{C}$,

$$\frac{(\ell \circ c)^{(n)}(z)}{n!} = \frac{\ell(c^{(n)}(z))}{n!} = \frac{1}{2\pi i}\int\limits_{|h|=1} \frac{\ell(c(hz))}{h^{n+1}} dh.$$

Thus, $\ell \circ c^{(n)}(\mathbb{D}) \subset \ell(n!b)$ (see Lemma 5.11). By the Hahn–Banach separation theorem (see Proposition 2.2) applied to $n!b$ and to $\{c^{(n)}(z)\}$ for any $z \in \mathbb{D}$, we get that $c^{(n)}(\mathbb{D}) \subset n!b$. □

Proposition 5.13. Let $c : \mathbb{C} \to E$ be a holomorphic curve. For any $z \in \mathbb{C}$, $c^{(n)}(z) \in E$ and c can be uniquely written as a series uniformly converging on bounded disks of $\mathbb{C}$:

$$c : z \mapsto \sum_n \frac{1}{n!} c^{(n)}(0) z^n.$$

Moreover, this series is Mackey-converging at each point of $\mathbb{C}$.

Proof. For every $\ell \in E'$, $\ell \circ c$ is a holomorphic function from $\mathbb{C}$ to $\mathbb{C}$. It does thus correspond in every point to its Taylor series in 0, and as $c^{(n)}(z) \in E$ for every z, we have

$$\ell \circ c(z) = \sum_n \frac{1}{n!} (\ell \circ c)^{(n)}(0) z^n = \sum_n \frac{1}{n!} \ell(c^{(n)}(0)) z^n = \ell(\sum_n \frac{1}{n!} c^{(n)}(0) z^n).$$

As E' is point separating, we have for every $z \in \mathbb{C}$:

$$c(z) = \sum_n \frac{1}{n!} c^{(n)}(0) z^n.$$

For any $r > 0$, the closed and absolutely convex closure b_r of $\{\frac{1}{n!} c^{(n)}(0) r^n \mid n \in \mathbb{N}\}$ is bounded. It is indeed weakly bounded as the power series $\sum_n \frac{1}{n!} \ell(c^{(n)}(0)) z^n$ converges uniformly on the open disk of centre 0 and radius r. Thus, for every $|z| < r$, we have

$$\sum_{n \geqslant N} \frac{1}{n!} c^{(n)}(0) z^n \in \sum_{n \geqslant N} \left(\frac{|z|}{r}\right) b_r \subset \left(\frac{|z|}{r}\right)^N \frac{1}{1 - \frac{|z|}{r}} b_r$$

and the series $\sum_n \frac{1}{n!} c^{(n)}(0) z^n$ does Mackey-converge towards $c(z)$.

□

Power series and holomorphy. The goal of this paragraph is to prove that power series, as presented in Definition 5.6, preserve holomorphic curves (see Theorem 5.16). This will show that they follow the same pattern as smooth functions that preserve smooth curves. As mentioned in Kriegl and Michor (1997, II.7.19.6), functions preserving holomorphic curves on $\mathbb{D}$ are locally power series, but we do not know if the preservation of holomorphic curves characterizes our power series.

The following property is adapted from Kriegl and Michor (1997, II.7.6).

Lemma 5.14. A holomorphic curve into E locally factors through a Banach space E_b generated by a bounded set $b \subset E$ (see Definition 2.7).

Proof. Consider c a holomorphic curve, $z \in \mathbb{C}$ and w a compact neighbourhood of z. Let us denote b the absolutely convex closed closure of $c(w)$. For any $\ell \in E'$, the Cauchy inequality (5.2) gives us for r small enough

$$\frac{r^k}{k!} (\ell \circ c)^{(k)}(z) \in \ell(b).$$

Thus, for z' close enough to z in $\mathbb{C}$,

$$(\ell \circ c)(z') = \sum_{k \geqslant 0} \left(\frac{z - z'}{r}\right)^k \frac{r^k}{k!} (\ell \circ c)^{(k)}(z) \ \in \ \sum_{k \geqslant 0} \left(\frac{z - z'}{r}\right)^k \ell(b).$$

Then, as E' is point separating, we get that

$$c(z') \in \sum_{k\geqslant 0} \left(\frac{z-z'}{r}\right)^k b.$$

And for z' close enough to z, $c(z') \in E_b$. □

Now, we want to show that for every holomorphic curve c, if $f : E \to F$ is a power series, then $f \circ c$ is also a holomorphic curve (see Theorem 5.16). This is a generalization of Kriegl and Michor (1997, II.7.17). Briefly, this is shown by working with E_b, so as to use Banach space properties. Remember that a space E is Mackey-complete if and only if each E_b is a Banach space (see Proposition 2.8).

Lemma 5.15. Let $f = \sum_k f_k$ be a power series from E to F. For any bounded set b of E, the set $\{f_k(x) \mid x \in b\}$ is bounded in F.

Proof. Let us write $S_n = \sum_{k\leqslant n} f_k$, and fix b any bounded set of E. Then, by definition of power series, S_n converges uniformly on bounded sets, hence for every U neighbourhood of 0 in E, there is p such that if $n, m \geqslant p$ we have $(S_n - S_m)(b) \subset U$. In particular, for $k \geqslant p+1$, $f_k(b) \subset U$. Because f_j is bounded for $j \leqslant p$, there is $\lambda_j \in \mathbb{C}$ such that $f_j(b) \in \lambda_j U$. Finally, we get $\{f_k \mid k \in \mathbb{N}\}(b) \subset \max\{1, \lambda_0, \dots, \lambda_p\}U$. □

Theorem 5.16. Power series send holomorphic curves to holomorphic curves.

Proof. Let $f = \sum_k f_k : E \to F$ be a power series, and $c : \mathbb{C} \to E$ be a holomorphic curve. Let $\tilde{f}_k$ be the k-linear bounded map associated to the k-monomial f_k.

Let us show that the curve $f \circ c : \mathbb{C} \to F$ is holomorphic. Thanks to Proposition 5.10, it is enough to show that for every $\ell \in F'$, $\ell \circ f \circ c : \mathbb{C} \to \mathbb{C}$ is holomorphic. Let us fix $z_0 \in \mathbb{C}$ and show that locally around z_0, $\ell \circ f \circ c$ is complex derivable. By translating c, we can assume w.l.o.g. that $c(z_0) = 0$, and $z_0 = 0$. Besides, by Proposition 5.14, we can assume w.l.o.g. that E is a Banach space.

Thanks to Propostion 5.13, we can write locally c as a Mackey-converging power series in E. For every $z \in \mathbb{C}$, we have

$$c(z) = \sum_n a_n z^n.$$

Moreover, this series converges uniformly on $\mathbb{D}$.

Because ℓ is linear and continuous, we have $\ell \circ f = \sum_k \ell \circ f_k$. Besides, for any $k \in \mathbb{N}$, $\ell \circ \tilde{f}_k$ is k-linear and bounded. Thanks to Lemma 5.3, $\sum_{n_1} \cdots \sum_{n_k} \ell \circ \tilde{f}_k(a_{n_1}, \dots, a_{n_k}) z^{n_1+\dots+n_k}$ converges to $\ell \circ f_k(c(z)) = \ell \circ \tilde{f}_k(c(z), \dots, c(z))$. We thus have

$$\ell \circ f(c(z)) = \sum_k \sum_{n_1} \cdots \sum_{n_k} \ell \circ \tilde{f}_k(a_{n_1}, \dots, a_{n_k}) z^{n_1+\dots+n_k}.$$

Let us now apply Lemma 5.15 to the unit disk U, which is bounded, in the Banach space E. We get that $\{\ell \circ \tilde{f}_k(x_1, \dots, x_k) \mid k \in \mathbb{N}, x_j \in U\}$ is bounded. Since $\sum_n a_n z^n$ converges, for any $|z| < 1$ and n big enough, $a_n z^n \in U$. Thus, for $r < 1$, we have for all $n \geqslant N$ $a_n r^n \in U$, thus the following set is bounded:

$$b = \left\{ \ell \circ \tilde{f}_k(a_{n_1} r^{n_1}, \ldots, a_{n_k} r^{n_k}) | n_i \geqslant N \right\}.$$

Following Kriegl and Michor (1997, II.7.17), consider z and r such that $|z| < \frac{1}{2}$ and $2|z| < r < 1$, then

$$\begin{aligned}
&\sum_k \sum_{n_1} \cdots \sum_{n_k} \ell \circ \tilde{f}_k(a_{n_1}, \ldots, a_{n_k}) z^{n_1+\cdots+n_k} \\
&\qquad = \sum_k \sum_{n_1} \cdots \sum_{n_k} \ell \circ \tilde{f}_k(a_{n_1} r^{n_1}, \ldots, a_{n_k} r^{n_k}) \frac{z^{n_1+\cdots+n_k}}{r^{n_1+\cdots+n_k}}, \\
&\qquad = \sum_n \sum_k \sum_{n_1+\cdots+n_k=n} \ell \circ \tilde{f}_k(a_{n_1} r^{n_1}, \ldots, a_{n_k} r^{n_k}) \frac{z^{n_1+\cdots+n_k}}{r^{n_1+\cdots+n_k}}.
\end{aligned} \tag{1}$$

Now, we look at the last sum and get

$$\sum_n \sum_k \sum_{n_1+\cdots+n_k=n} \ell \circ \tilde{f}_k(a_{n_1} r^{n_1}, \ldots, a_{n_k} r^{n_k}) \frac{z^{n_1+\cdots+n_k}}{r^{n_1+\cdots+n_k}} \in \sum_n (2^n - 1) \left(\frac{z}{r}\right)^n b.$$

This is an absolutely converging sum, and the permutation of the sums in Equation (1) is justified by Fubini's theorem. Finally, $\ell \circ f \circ c$ is holomorphic in $\mathbb{C}$, as it is the sum of an absolutely converging power series. □

Another proof of this theorem uses Hartog's theorem (Kriegl and Michor 1997, II.7.9), and the fact that a bounded k-monomial sends a holomorphic curve to a holomorphic curve.

Lemma 5.17. Let $f = \sum_k f_k$ be a power series between E and F. Then, for every $x \in E$ and $n \in \mathbb{N}$, $c : z \mapsto f(zx)$ is a holomorphic curve into F whose n-th derivative in 0 is $n! f_n(x)$.

Proof. The curve $c : z \mapsto f(zx)$ is holomorphic thanks to Theorem 5.16. Since the scalar multiplication on E is continuous, the set $\{zx \mid |z| < 1\}$ is bounded. By Definition 5.6 of power series, $\sum_k f_k(zx) = \sum_k f_k(x) z^k$ converges uniformly on the unit disk $\mathbb{D}$ of $\mathbb{C}$. Thanks to the uniqueness of the decomposition (see Lemma 5.13), its nth derivative is $n! f_n(x)$. □

Corollary 5.18. The k-monomials in the development of a power series are unique.

Proposition 5.19. Every power series $f \in S(E, F)$ verifies a Cauchy inequality: if b is an absolutely convex set in E and if b' is an absolutely convex and closed set in E such that $f(b) \subset b'$, then for all $n \in \mathbb{N}$ we have also

$$f_n(b) \subset b'.$$

Proof. For every $x \in E$, $c : z \mapsto f(zx)$ is a holomorphic curve into F whose nth-derivative is $n! f_n(x)$ by Lemma 5.17. For every $\ell \in F'$, $\ell \circ c$ is holomorphic and satisfies a Cauchy Formula:

$$\tfrac{1}{n!}(\ell \circ c)^{(n)}(0) = \ell(\frac{c^{(n)}(0)}{n!}) = \ell(f_n(x)) = \frac{1}{2i\pi} \int_{|h|=1} \frac{\ell(f(hx))}{h^{n+1}} dh.$$

As b is absolutely convex, we conclude thanks to the Hahn–Banach separation theorem (see Proposition 2.2) that for every $x \in b$, for every $n \in \mathbb{N}$, $f_n(x) \in b'$ (see Lemma 5.11). □

5.3. *Convergence of power series*

Thanks to the Cauchy inequality, we will show the Mackey-convergence of the partial sums of a power series. This property is fundamental in the construction of the cartesian closed category of Mackey-complete spaces and power series. It will allow for example to ensure that when composing a bounded function with a power series, the bounded function distributes over the sum of monomials (see Proposition 5.25).

Proposition 5.20. If $f = \sum_n f_n$ is a power series, then its partial sums Mackey-converge towards f in $\mathcal{B}(E, F)$.

Proof. Let b be an absolutely convex and bounded subset of E and b' be the absolutely convex and closed closure of $f(b)$. By Proposition 5.19, for all $n \in \mathbb{N}$, $f_n(b) \subset b'$. As f_n is n-homogeneous, we also have $f_n(\frac{1}{2}b) \subset \frac{1}{2^n}b'$.

If B denotes the equibounded set $\{f \in \mathcal{B}(E,F) \mid f(\frac{1}{2}b) \subset b'\}$, then $f \in B$ as $\frac{1}{2}b \subset b$, $f_0 \in B$ and for every n, $f_n \in \frac{1}{2^n}B$. Thus, partial sums do Mackey-converge towards f, as

$$\forall N \in \mathbb{N},\ f - \sum_{n=0}^{N} f_n \in \sum_{n>N} \tfrac{1}{2^n} B.$$

□

Our definition of power series allows us to make nice connection between their weak, strong, and simple convergence. This will allow us to prove the cartesian closedeness of the category of Mackey-complete spaces and power series between them.

Proposition 5.21. Let $\{f_k \mid k \in \mathbb{N}\}$ be a family of k-monomials from E to F. If for every $\ell \in F^\times$ (resp. $\ell \in F'$) and $x \in E$, $\sum_k \ell \circ f_k(x)$ converges in $\mathbb{C}$, then for any $x \in E$, $\sum_k f_k(x)$ converges in F.

Proof. Let us fix $x \in E$. By assumption, for any $\ell \in F'$, $\sum_k \ell \circ f_k(2x)$ converges in $\mathbb{C}$, so $\{\ell \circ f_k(2x) \mid k \in \mathbb{N}\}$ is bounded in $\mathbb{C}$. By Proposition 2.1 (resp. by the Mackey–Ahrens Theorem), $\{f_k(2x) \mid k \in \mathbb{N}\}$ is bounded in F, its closure denoted b' is also bounded. We get that, for all $N \in \mathbb{N}$,

$$\sum_{k \geqslant N} f_k(x) \subset \sum_{k \geqslant N} \tfrac{1}{2^k} b'.$$

Hence, $\sum_k f_k(x)$ is Mackey–Cauchy and so converges in F. □

Proposition 5.22. Let $f : E \to F$ be a bounded function and let f_k be k-monomials such that for every $\ell \in F'$, $\sum_k \ell \circ f_k$ converges towards $\ell \circ f$ uniformly on bounded sets of E. Then, $f = \sum_k f_k$ is also a power series.

Proof. Let b be a bounded set and b' be the absolutely convex and closed closure of $f(2b)$. For any $\ell \in F'$, since $\ell \circ f$ is a power series, it satisfies a Cauchy Inequality thanks to Proposition 5.19 (notice that $\ell(b')$ is absolutely convex and that $(\ell \circ f)(2b) \subset \ell(b')$). Therefore, for any $k \in \mathbb{N}$, $(\ell \circ f_k)(2b) \subset \ell(b')$. By the Hahn–Banach separation theorem (see Proposition 2.2) and since f_k is k-linear, we get that $f_k(b) \subset \frac{1}{2^k} b'$. Thus, $\sum_k f_k$ Mackey-converges uniformly to f on bounded sets of E. Since Mackey-convergence entails convergence, we get that $f = \sum_k f_k$ is a power series. □

The two last propositions helped us to infer strong convergence from weak convergence, the following will allow us to deduce uniform convergence from pointwise convergence.

Proposition 5.23. Let $\sum_k f_k : E \to F$ be a series of k-monomials. If the sum converges pointwise towards a bounded function $f : E \to F$, then f is a power series.

Lemma 5.24. Consider E a Fréchet space and for every $k \in \mathbb{N}$ $f_k \in \mathcal{L}^k(E, \mathbb{C})$. Then, $\sum f_k$ converges pointwise on E if and only if it converges uniformly on every bounded set of E.

Proof. (see Kriegl and Michor (1997, II.7.14) for details) The reverse implication is straightforward. Let us sketch the proof of the direct implication, and suppose $\sum f_k$ converges pointwise. We want to show that $\{\tilde{f}_k(x_1, \dots, x_k) \mid k \in \mathbb{N}, x_i \in U\}$ is bounded on a certain 0-neighbourhood U. If this is true, then $\sum f_k$ converges uniformly on λU for $\lambda < 1$, and thus on every bounded set of E. To do so, we consider the closed sets $A_{K,r} = \{x \in E \mid \forall k \in \mathbb{N},\ |f_k(x^k)| \leqslant K r^k\}$. By Baire's theorem, there is such an $A_{K,r}$ whose interior is non-empty. The polarization formula then helps to conclude. □

Proof of Proposition 5.23 Let us fix $\ell \in F'$. For every $x \in E$, $\sum_k \ell \circ f_k(x)$ converges towards $\ell \circ f(x)$ in $\mathbb{C}$, and $\ell \circ f$ is bounded. Let b be a bounded set. Then, according to Lemma 5.24 which relates pointwise convergence and uniform convergence of power series on Banach spaces, the power series $\sum_k \ell \circ f_k(x)$ converges uniformly on E_b (as it is a Banach space, see Proposition 2.8), hence on b. We have proved that $\sum_k f_k$ converges weakly uniformly on bounded sets. By Proposition 5.22, we know that it converges (strongly) uniformly on bounded subsets. □

Proposition 5.25. Let ℓ be a linear bounded function from F to G and $f = \sum_k f_k$ be a power series from E to F. Then, $\ell \circ f$ is a power series and $\ell \circ f = \sum_n \ell \circ f_n$.

Proof. According to Proposition 5.20, there is a sequence of scalars (λ_n) decreasing towards 0 and a bounded set $B \subset S(E, F)$ such that, for all n,

$$f - \sum_{k=0}^{n} f_k \in \lambda_n B.$$

Thus, for every n, $\ell \circ f - \sum_{k=0}^{n} \ell \circ f_k \in \lambda_n \ell(B)$. Thus, applying this equation to every $x \in E$, we get that the partial sums of $\sum_k \ell \circ f_k(x)$ Mackey-converge towards $\ell \circ f(x)$. As $\ell \circ f$ is a bounded function, we have that $\ell \circ f$ is a power series thanks to Proposition 5.23. □

5.4. *A cartesian closed category*

Definition 5.26. Let us denote as $S(E, F)$ the space of all **power series** between E and F. We endow it with the topology of uniform convergence on bounded subsets of E. The bounded sets resulting from this topology are the equibounded sets of functions.

We write **Series** the category of Mackey-complete spaces and power series. Holomorphic maps, as defined in Kriegl and Michor (1997) are in particular smooth (Kriegl and Michor 1997, II.7.19.8). Thus, according to Proposition 5.16, power series as defined here are smooth.

Proposition 5.27. We have a bounded inclusion of $S(E, F)$ into $\mathcal{C}^\infty(E, F)$.

Proof. Let B be a bounded set in $S(E, F)$. Let us prove that B is bounded in $\mathcal{C}^\infty(E, F)$, i.e. for every smooth curve $c \in \mathcal{C}_E$, every bounded set $b \subset \mathbb{R}$ and every $j \in \mathbb{N}$, the following set is bounded in F:

$$\{(f \circ c)^{(j)}(x) \mid f \in B,\ x \in b\}.$$

Let us fix $c \in \mathcal{C}_E$ and $j \in \mathbb{N}$.

Let C_j be the set made of c and its derivatives of order at most j. Since c and its up to jth derivatives are smooth, they are bounded and send b to a common absolutely convex bounded set b' of E, i.e. $C_j(b) \subset b'$.

As a power series $f = \sum_n f_n$ converges uniformly on bounded sets of E, we can derivate under the sum. Thus, $(f \circ c)^{(j)}(x) = \sum_n (f_n \circ c)^{(j)}(x)$. It is possible to show by induction on j that $(f_n \circ c)^{(j)}(x) = \left(\tilde{f}_n(c(\cdot), \ldots, c(\cdot))\right)^{(j)}(x) = \sum_{l=1}^{j^n} \alpha_j^l \tilde{f}_n(c_1^\ell(x), \ldots, c_n^\ell(x))$ with $c_k^\ell \in C_j$ and $\alpha_j^\ell \leqslant n^j$ an integer, where $\tilde{f}_n$ is the symmetric n-linear map from which f_n results. Therefore, we have

$$(f_n \circ c)^{(j)}(b) \subset n^j j^n \tilde{f}_n(b').$$

Now, let $b_E = 8j\, b'$. According to Proposition 5.19, as $f(b_E) \subset B(b_E)$, we get

$$f_n(b') \subset \tfrac{1}{(8j)^n} B(b_E).$$

Thanks to the polarization formula (see Lemma 5.2), for any $x_1, \ldots, x_n \in b'$,

$$\tilde{f}_n(x_1, \ldots, x_n) = \tfrac{1}{n!} \sum_{\epsilon_1, \ldots, \epsilon_n = 0}^{1} (-1)^{n - \sum_{j=1}^{n} \epsilon_j} f_n\left(\sum_{j=1}^{n} \epsilon_j x_j\right).$$

Then, for any $\ell \in F^{\times}$, we get

$$\begin{aligned}\left|\ell \circ \tilde{f}_n(x_1,\dots,x_n)\right| &\leqslant \frac{1}{n!}\sum_{\epsilon_1,\dots,\epsilon_n=0}^{1}\left|\ell \circ f_n\left(\sum_{j=1}^{n}\epsilon_j x_j\right)\right| \\ &= \frac{1}{n!}\sum_{\epsilon_1,\dots,\epsilon_n=0}^{1}\left(\textstyle\sum_{i=1}^{n}\epsilon_i\right)^n \left|\ell \circ f_n\left(\frac{\sum_{j=1}^{n}\epsilon_j x_j}{\sum_{j=1}^{n}\epsilon_j}\right)\right|\end{aligned}$$

Note that in the last sum, $\sum_j \epsilon_j$ can be supposed to be strictly positive, as when all ϵ_j equals 0 then $\left|\ell \circ f_n\left(\sum_{j=1}^{n}\epsilon_j x_j\right)\right| = 0$. Now, there is exactly $\binom{n}{j}$ ways of having $\sum_{i=1}^{n}\epsilon_i = j$:

$$\left|\ell \circ \tilde{f}_n(x_1,\dots,x_n)\right| \leqslant \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j} j^n \left|\ell \circ f_n\left(\frac{\sum_{j=1}^{n}\epsilon_j x_j}{\sum_{j=1}^{n}\epsilon_j}\right)\right|$$

Consider the binomial formula $(1+x)^n = \sum_{j=0}^{n}\binom{n}{j}x^j$. If we differentiate this expression and we multiply the result by x, we get

$$nx(1+x)^{n-1} = \sum_{j=0}^{n} j\binom{n}{j}x^j.$$

By repeating this operation $(n-1)$ times, we get

$$\sum_{k=1}^{n} n\dots(n-k+1)\binom{n}{k}(1+x)^{n-k}x^k = \sum_{j=0}^{n} j^n\binom{n}{j}x^j.$$

Taking $x = 1$ thus implies $\sum_{k=1}^{n}\frac{n!}{(n-k)!}\binom{n}{k}2^{n-k} = \sum_{j=0}^{n}\binom{n}{j}j^n$. We have then

$$\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}j^n = \sum_{k=0}^{n-1}\frac{1}{(k)!}\binom{n}{k}2^k \leqslant 2^n\sum_{k=0}^{n-1}\frac{1}{(k)!}\binom{n}{k} \leqslant 2^n\sum_{k=0}^{n-1}\binom{n}{k} \leqslant 2^{2n}.$$

Therefore,

$$\begin{aligned}\left|\ell \circ \tilde{f}_n(x_1,\dots,x_n)\right| &\leqslant 4^n\left|\ell \circ f_n\left(\frac{\sum_{j=1}^{n}\epsilon_j x_j}{\sum_{j=1}^{n}\epsilon_j}\right)\right| \\ &\leqslant 4^n\frac{1}{(8j)^n}|\ell \circ B(b_E)| \\ &\leqslant \frac{1}{(2j)^n}|\ell \circ B(b_E)|\end{aligned}$$

Thanks to Lemma 2.1, $b_F = (2j)^n\left\{\tilde{f}_n(x_1,\dots,x_n) \mid \forall f \in B,\ \forall x_1,\dots,x_n \in b'\right\}$ is bounded.

To conclude, for every $f \in B$,

$$(f_n \circ c)^{(j)}(b) \subset n^j j^n \tilde{f}_n(b') \subset \frac{n^j j^n}{(2j)^n}b_F \subset \frac{n^j}{2^n}b_F,$$

so that,

$$(f \circ c)^{(j)}(b) \subset \sum_n (f_n \circ c)^{(j)}(b) \subset \sum_n \frac{n^j}{2^n} b_F.$$

□

Let us note that any subset of $S(E,F)$ which is the restriction to $S(E,F)$ of a bounded set of $\mathcal{C}^\infty(E,F)$ is uniformly bounded. Indeed, according to Kriegl and Michor (1997, 4.4.7), the bornology on $\mathcal{C}^\infty(E,F)$ is the coarsest one making all pointwise evaluation $ev_x : \mathcal{C}^\infty(E,F) \to F$ bounded. But when we artificially consider on $\mathcal{C}^\infty(E,F)$ the bornology of all uniformly bounded set, all pointwise evaluation are bounded. So this bornology is finer than the one resulting from the topology of $\mathcal{C}^\infty(E,F)$, that is bounded sets of $\mathcal{C}^\infty(E,F)$ are uniformly bounded.

Proposition 5.28. When F is Mackey-complete so is $S(E,F)$.

Proof. Consider $(f_\gamma)_{\gamma\in\Gamma}$ a Mackey–Cauchy net in $S(E,F)$. There is a positive real net $(\lambda_{\gamma,\gamma'})_{\gamma,\gamma'\in\Gamma}$ converging towards 0 and an equibounded set B in $S(E,F)$ such that

$$f_\gamma - f_{\gamma'} \in \lambda_{\gamma,\gamma'} B. \tag{2}$$

We can suppose w.l.o.g. that $B = \{f \mid \forall b \text{ bounded in } E,\ f(b) \subset B(b)\}$ and that B is absolutely convex and closed.

For all $x \in E$, $B(\{x\})$ is bounded in F, and $(f_\gamma(x))_{\gamma\in\Gamma}$ is a Mackey–Cauchy net in F. Since F is Mackey-complete, for each $x \in E$, $f_\gamma(x)$ converges towards $f(x)$ in F.

Let us show that $f : E \to F$ is a power series. Since $f_\gamma \in S(E,F)$, we can write $f_\gamma = \sum_n f_{\gamma,n}$. Now, we fix $n \in \mathbb{N}$ and prove that

$$f_{\gamma,n} - f_{\gamma',n} \in \lambda_{\gamma,\gamma'} B.$$

From Equation (2), we have that, for any b absolutely convex and bounded in E, $(\sum_n f_{\gamma,n} - f_{\gamma',n})(b) \in \lambda_{\gamma,\gamma'} B(b)$. Thus, by Proposition 5.19, for all $n \in \mathbb{N}$, $(f_{\gamma,n} - f_{\gamma',n})(b) \in \lambda_{\gamma,\gamma'} B(b)$. We conclude by our assumption on the shape of B.

Then, $(f_{\gamma,n})_{\gamma\in\Gamma}$ is a Mackey–Cauchy net in $\mathcal{L}^n(E,F)$, which is Mackey-complete according to Proposition 5.5. Thus, $(f_{\gamma,n})_{\gamma\in\Gamma}$ converges in $\mathcal{L}^n(E,F)$ and we denote as f_n its limit.

Let us show that $\sum_n f_n$ converges pointwise towards f. Let us fix $x \in E$ and V an absolutely convex neighbourhood of 0 in F. We denote as $\mathbb{D}x$ the set $\{zx \mid z \in \mathbb{C}, |z| < 1\}$. We will show that each part of the following expression is small enough:

$$f(x) - \sum_{n<N} f_n(x) = \left(\lim_{\gamma'\to\infty} f_{\gamma'}(x) - f_\gamma(x)\right) + \left(f_\gamma(x) - \sum_{n<N} f_{\gamma,n}(x)\right) + \sum_{n<N} \left(f_{\gamma,n}(x) - f_n(x)\right).$$

Since $2\mathbb{D}x$ is bounded, then so is $B(2\mathbb{D}x)$ and there is $\mu > 0$ such that $B(2\mathbb{D}x) \subset \mu V$. Let $\gamma_0 \in \Gamma$ be such that when $\gamma, \gamma' \geqslant \gamma_0$, we have $|\lambda_{\gamma,\gamma'}\mu| < 1$, and so $B(\mathbb{D}x) \subset \lambda_{\gamma,\gamma'} B(2\mathbb{D}x) \subset V$. Then,

$$\forall \gamma', \gamma \geqslant \gamma_0,\ f_{\gamma'}(x) - f_\gamma(x) \in \lambda_{\gamma,\gamma'} B(\mathbb{D}x) \quad \text{and} \quad \lim_{\gamma'\to\infty} f_{\gamma'}(x) - f_\gamma(x) \in V.$$

By convergence, for $N \in \mathbb{N}$ big enough,

$$f_\gamma(x) - \sum_{n<N} f_{\gamma,n}(x) \in V.$$

Moreover, for every n, we have $f_{\gamma,n}(2x) - f_{\gamma',n}(2x) \in \lambda_{\gamma,\gamma'} B(2\mathbb{D}x)$, and since they are n-monomials, $f_{\gamma,n}(2x) - f_{\gamma',n}(2x) = 2^n(f_{\gamma,n}(x) - f_{\gamma',n}(x))$. Finally, by taking the limit $\gamma' \to \infty$, we get

$$f_{\gamma,n}(x) - f_n(x) \in \frac{1}{2^n}\overline{V}.$$

To sum up, $\sum_n f_n$ converges pointwise towards f, for N big enough,

$$f(x) - \sum_{n<N} f_n(x) \in V + V + \left(\sum_{n<N} \frac{1}{2^n}\right)\overline{V} \subset 5V.$$

Now, we apply Proposition 5.23, to show that $\sum f_k$ does converge uniformly on bounded sets of E towards f and therefore $f \in S(E, F)$. It is sufficient to show that f is bounded since we have already shown the simple convergence. Let b be an absolutely convex and bounded set b of E. Consider $\gamma \in \Gamma$. Then, we get

$$f(x) = f_\gamma(x) + (f(x) - f_\gamma(x)) = f_\gamma(x) + \lim_{\gamma' \to \infty} \sum_n (f_{\gamma',n}(x) - f_{\gamma,n}(x)).$$

If M is an upper bound of the net $(\lambda_{\gamma,\gamma'})$, we get that $f(b) \subset f_\gamma(b) + M\overline{B(b)}$. □

In order to prove that the composite of two power series is also a power series, we need to use Fubini's theorem and permute sums. We will have to embed our series in $\mathbb{C}$ and to use Propositions 5.21 and 5.23 that relates weak, strong, pointwise, and uniform convergences.

Theorem 5.29. The composition of two power series is a power series.

Proof. Consider $f = \sum_n f_n : E \to F$ and $g = \sum_k g_k : G \to E$ two power series. Let us show that $f \circ g : G \to F$ is a sum $\sum_m h_m$ of m-monomials converging uniformly on bounded sets of G. Let us use $\tilde{f}_n$ (resp. $\tilde{g}_k$) for the n-linear (resp. k-linear) function corresponding to f_n (resp. g_k).

Because the series $\sum_k g_k$ Mackey-converges (see Proposition 5.20), and because, for each $n \in \mathbb{N}$, f_n is an n-monomial, we have

$$\forall x \in G, \tilde{f}_n(g(x)) = \sum_{k_1,\ldots k_n \geqslant 0} f_n(g_{k_1}(x), \ldots, g_{k_n}(x)).$$

Notice that $\tilde{f}_n(g_{k_1}(x), \ldots, g_{k_n}(x))$ is a $(k_1 + \cdots + k_n)$-monomial in x.

Let us write

$$h_m : x \mapsto \sum_{n \geqslant 0} \sum_{\substack{k_1+\cdots+k_n=m \\ k_i \geqslant 0}} \tilde{f}_n(g_{k_1}(x), \ldots, g_{k_n}(x)) \tag{3}$$

and show that h_m is a well defined bounded m-monomial such that $f \circ g = \sum_m h_m$.

Let us consider $x \in G$ and fix $\ell \in F'$. The power series

$$\ell \circ f \circ g = \sum_{k_1,\ldots,k_n \geqslant 0} \ell(\tilde{f}_n(g_{k_1}(x), \ldots, g_{k_n}(x))) \tag{4}$$

is convergent on $3\mathbb{D}x$, hence absolutely convergent on $2\mathbb{D}x$, where $\mathbb{D}$ stands for the unit ball in $\mathbb{C}$. Thus, we can permute coefficients in the converging sum above. Therefore, the general term $\ell \circ h_m(x)$ of the series $\sum_{m \geqslant 0} \ell \circ h_m(x)$, which is obtained from Equation (4) by permuting indices of the sum, is also the sum of an absolutely converging series. By Proposition 5.21, since for any $\ell \in F'$ and any $x \in G$, $\ell \circ h_m(x)$ is the limit of a converging sum in $\mathbb{C}$, thus for any $x \in E$, $h_m(x)$ is well-defined in F. Moreover, for any $\ell \in F'$, we have proved that $\ell \circ f \circ g(x) = \sum_{m \geqslant 0} \ell \circ h_m(x) = \ell \circ \sum_{m \geqslant 0} h_m(x)$, so by the Hahn–Banach separation theorem (see Proposition 2.2):

$$\forall x \in G, f \circ g(x) = \sum_m h_m(x).$$

Let b be a bounded set in G. Since g is bounded, $g(2b)$ is a bounded set in E, and we set b' to be its absolutely convex and closed closure which is also bounded. Let b'' be the absolutely and closed closure of the bounded set $f(2g(2b))$ of F. Now, by Proposition 5.19, if $x \in b$, then $g_k(2x) \in b'$ and $\tilde{f}_n(2g_{k_1}(2x), \ldots, 2g_{k_n}(2x)) \in b''$. Since g_{k_i} and f_n are monomials, for $x \in b$, we get $g_{k_i}(x) \in \frac{1}{2^{k_i}} b'$ and $\tilde{f}_n(g_{k_1}(x), \ldots, g_{k_n}(x)) \in \frac{1}{2^n} \frac{1}{2^{\sum k_i}} b''$. Since there is exactly $\binom{m+n-1}{m}$ ways of choosing n natural numbers whose sum is m, we get from Formula (3):

$$h_m(x) \in \frac{1}{2^m} \sum_n \binom{m+n-1}{m} \frac{1}{2^n} b'.$$

Moreover, we have

$$\lim_{n \to \infty} \tfrac{m!}{n^m} \binom{m+n-1}{m} = 1.$$

Thus, $\sum_n \binom{m+n-1}{m} \frac{1}{2^n}$ is absolutely converging. We have

$$h_m(b) \subset \sum_n \binom{m+n-1}{m} \frac{1}{2^n} b'$$

so h_m is bounded. As it is a converging sum of m-monomials, h_m is also an m-monomial.

We conclude that $f \circ g$ is a power series by Proposition 5.23, as $\sum_m h_m$ is a series of bounded m-monomials pointwise converging to $f \circ g$ which is also bounded. □

We can finally address the problem of cartesian closedeness, which is solved by getting back to the scalar case and by using Fubini's theorem.

Theorem 5.30. If E, F, and G are Mackey-complete spaces, then there are natural isomorphisms:

$$S(E, S(F, G)) \simeq S(E \times F, G).$$

Proof. Let us first notice that if the stated isomorphisms is true as an equality between sets, then the topologies on these spaces are the same. Indeed, sending $B_1 \times B_2$ on a weak 0-neighbourhood U is equivalent to sending B_1 to a function which will send B_2 to U. This will give us a homeomorphism, thus a bounded isomorphism, between the two spaces.

Let us define the two maps inverse of one another, as shown by direct computation:

$$\phi : \begin{cases} S(E \times F, G) \to S(E, S(F, G)) \\ \sum_k f_k \mapsto \left(x \mapsto \left(y \mapsto \sum_n \sum_m \binom{n+m}{n} \tilde{f}_{n+m}(\underbrace{(x,0),\dots,(x,0)}_{n \text{ times}}, \overbrace{(0,y),\dots,(0,y)}^{m \text{ times}}) \right) \right) \end{cases},$$

and

$$\psi : \begin{cases} S(E, S(F, G)) \to S(E \times F, G) \\ \sum_n (f_n : x \mapsto \sum_m f^x_{n,m}) \mapsto \left((x, y) \mapsto \sum_k \sum_{n+m=k} f^x_{n,m}(y) \right) \end{cases}.$$

We need to show that they are well defined, linear, bounded, and natural in E, F, and G. The difficulty is in showing that their image is indeed made of power series. We will do it on ψ, the proof for ϕ using similar tools and being easier.

Consider a function $f \in S(E, S(F, G))$. It can be written as $\sum_n (f_n : x \mapsto \sum_m f^x_{n,m})$, each f_n being a bounded n-monomial from E to $S(F, G)$, and each $f^x_{n,m}$ being a bounded m-monomial from F to G. The function $(x, y) \mapsto \sum_{n+m=k} f^x_{n,m}(y)$ is a bounded k-monomial.

Let us fix $\ell \in G^\times$, $y \in F$ and define $\chi^y : S(F, G) \to \mathbb{C}$, $g \mapsto \ell \circ g(y)$. If $\mathcal{B}$ is bounded in $S(F, G)$, then $\mathcal{B}(y)$ is bounded in G and $\chi^y(g)$ is bounded in $\mathbb{C}$, hence $\chi^y \in S(F, G)^\times$. Moreover, because f is a power series, we know from Proposition 5.20 that its partial sums are Mackey-convergent and from Proposition 2.6 that χ^y preserves Mackey-convergence. Thus, for any $x \in E$, we have that

$$\sum_n \chi^y \left(\sum_m f^x_{n,m} \right) = \sum_n \sum_m l \circ f^x_{n,m}(y).$$

In particular, let us fix x and y, then $\sum_n \sum_m \ell \circ f^{2x}_{n,m}(2y)$ Mackey-converges in $\mathbb{C}$. Therefore, $\ell \circ f^{2x}_{n,m}(2y) = 2^n 2^m l \circ f^x_{n,m}(y)$ is the general term of a bounded double sequence and the radius of convergence of the $\mathbb{C}$-power series $\sum_n \sum_m \ell \circ f^x_{n,m}(y) z^{n+m}$ is at least 2. Finally, $\sum_n \sum_m \ell \circ f^x_{n,m}(y)$ converges absolutely in $\mathbb{C}$. Thanks to Fubini's theorem, we know that we can permute absolutely converging double series in $\mathbb{C}$. Then, $\sum_k \sum_{n+m=k} \ell \circ f^x_{n,m}(y)$ converges and is equal to $\sum_n \sum_m \ell \circ f^x_{n,m}(y)$. Thanks to Proposition 5.21, for any $x \in E$ and $y \in F$, $\psi(f)(x, y) \in G$, that is $\psi(f)$ is pointwise convergent.

We now prove that $\psi(f)$ converges uniformly on bounded subsets of E. First, notice that $\psi(f)$ is bounded. Indeed, f is bounded thanks to Proposition 5.7, and $\psi(f)$ sends $B_1 \times B_2$ to $f(B_1)(B_2)$. Proposition 5.23 states that a pointwise converging power series which converges towards a bounded function converges uniformly on bounded subsets of its codomain. We conclude that $\psi(f) \in S(E \times F, G)$.

The naturality of ψ in E and F resumes to precomposition. The naturality of ψ in G is proved by considering the fact that we postcompose ψ by a bounded linear function, which commutes to the sum of the power series which Mackey-converge. □

5.5. *From* **Lin** *to* **Series**

So far, we have proven that the category **Lin** of Mackey-complete spaces and bounded linear maps is symmetric monoidal closed and cartesian (see Section 3). We have also proven that the category **Series** of Mackey-complete spaces and power series is cartesian closed (see Section 5.4). We will now prove that there is a linear–non-linear adjunction between **Lin** and **Series** that comes from an exponential modality constructed exactly as presented in Section 4.2 for convenient spaces (see Blute et al. (2012) and Frölicher and Kriegl (1988, 5.1.1)).

Definition 5.31. Let E be a Mackey-complete space. For any $x \in E$, the **Dirac delta distribution** δ can be seen as a function on power series:

$$\delta : \begin{cases} E \rightarrow S(E, \mathbb{C})^\times \\ x \mapsto \delta_x : f \mapsto f(x) \end{cases}$$

δ is linear and bounded as it acts on bounded functions (see Proposition 5.7).

Exponential modality. For any Mackey-complete space E, we construct a Mackey-complete space $!E$ from $\delta(E)$ by applying the Mackey-completion procedure described in Proposition 2.9.

Definition 5.32. Let us use $!E$ for the Mackey-completion of the linear span of $\delta(E)$ in $S(E, \mathbb{C})^\times$ endowed with the topology of uniform convergence on bounded subsets of $S(E, \mathbb{C})$.

Thanks to Mackey-completion, in order to define a linear function on $!E$, it is sufficient to define it on δ_x for any $x \in E$. Let $f \in \mathcal{L}(E, F)$ be a bounded linear map. We define $!f : !E \rightarrow !F$ as the linear extension of

$$!f : \begin{cases} \delta(E) \rightarrow !F \\ \delta_x \mapsto \delta_{f(x)} \end{cases}$$

This function is linear by construction. Let us check that it is bounded. If $\mathcal{B}$ is an equibounded set in $S(E, \mathbb{C})^\times$, then $\{\delta_{f(x)} \mid \delta_x \in B\}$ is equibounded. Indeed, if B is bounded in $S(E, \mathbb{C})$, then

$$\{\delta_{f(x)}(B) \mid \delta_x \in \mathcal{B}\} = \{B(\{f(x)\}) \mid \delta_x \in \mathcal{B}\} = \{\delta_x(B \circ f)\} \mid \delta_x \in \mathcal{B}\}$$

is bounded, as f bounded makes $B \circ f = \{g \circ f \mid g \in B\}$ bounded. Hence, $!f$ is well defined, and is a bounded linear function. So we have indeed $!f \in \mathcal{L}(!E, !F)$.

Definition 5.33. We write $! : \textbf{Lin} \rightarrow \textbf{Lin}$ for the functor sending a Mackey-complete space E to $!E$, and a bounded linear map $f \in \mathcal{L}(E, F)$ to $!f \in \mathcal{L}(!E, !F)$.

Proposition 5.34. The functor $!$ is an exponential modality:

1. $(!, \rho, \epsilon)$ is a comonad, with

$$\epsilon_E : \begin{cases} !E \rightarrow E \\ \delta_x \mapsto x \end{cases} \qquad \rho_E : \begin{cases} !E \rightarrow !!E \\ \delta_x \mapsto \delta_{\delta_x} \end{cases}.$$

2. $! : (\mathbf{Lin}, \times, \top) \to (\mathbf{Lin}, \hat{\otimes}, 1)$ is a strong and symmetric monoidal functor, with

$$m^0 : \begin{cases} 1 \to !\top = !\{0\} \\ 1 \mapsto \delta_0 \end{cases} \qquad m^2_{E,F} : \begin{cases} !E \hat{\otimes} !F \to !(E \times F) \\ \delta_x \otimes \delta_y \mapsto \delta_{(x,y)} \end{cases}.$$

3. The following diagram commute:

$$\begin{tikzcd} !E \hat{\otimes} !F \arrow[r, "m^2_{E,F}"] \arrow[d, "\rho_E \hat{\otimes} \rho_F"'] & !(E \times F) \arrow[r, "\rho_{E\times F}"] & !!(E \times F) \arrow[d, "!\langle !\pi_1, !\pi_2 \rangle"] \\ !!E \hat{\otimes} !!F \arrow[rr, "m^2_{!E,!F}"'] & & !(!E \times !F) \end{tikzcd}$$

Proof. Notice that the natural transformations ϵ, ρ, and m^2 are defined by linearity and Mackey-complete extension. Then, it is enough to check the diagrams for comonad and symmetric monoidality on Dirac delta distributions. The morphisms m^0 and $m^2_{E,F}$ are natural isomorphisms with inverse

$$(m^0)^{-1} : \begin{cases} !\top = !\{0\} \to 1 \\ \delta_0 \mapsto 1 \end{cases} \qquad (m^2_{E,F})^{-1} : \begin{cases} !(E \times F) \to !E \hat{\otimes} !F \\ \delta_z \mapsto \delta_{\pi_1 z} \otimes \delta_{\pi_2 z} \end{cases}.$$

□

Distributions. The distribution space $S(E, \mathbb{C})^\times$ is equipped with a convolution product defined as follows. Notice that when restricted to $!E$, the convolution product can be obtained from the cartesian structure of **Lin** and from m^2.

Proposition 5.35. For any D_1 and D_2 in $S(E, \mathbb{C})^\times$, the convolution $D_1 * D_2$ is in $S(E, \mathbb{C})^\times$ and acts on $f \in S(E, \mathbb{C})$ as

$$(D_1 * D_2)f = D_1(x \mapsto (D_2(y \mapsto f(x + y)))).$$

Moreover, if D_1 and D_2 are in $!E$, then $D_1 * D_2$ is in $!E$.

Proof. Let $f \in S(E, \mathbb{C})$ and $x \in E$. Since $(x, y) \mapsto x + y$ is linear and bounded (and so a power series), the function $(x, y) \mapsto f(x + y)$ is a power series. Then, by cartesian closedness (Theorem 5.30), $x \mapsto (y \mapsto f(x + y)) \in S(E, S(E, \mathbb{C}))$. Since D_2 is bounded and linear, we get by post-composition that $x \mapsto D_2(y \mapsto f(x + y)) \in S(E, \mathbb{C})$; thus, we can apply D_1 to compute $(D_1 * D_2)f$. Notice that $D_1 * D_2$ is linear and bounded since all the involved operations are both bounded and linear.

Let D_1 and D_2 be in $!E$. Then, the convolution operator $*$ is the morphism:

$$\begin{tikzcd} !E \hat{\otimes} !E \arrow[r, "m^2_{E,E}"] & !(E \times E) \arrow[r, "!((x,y)\mapsto x+y)"] & !E \\ \delta_x \otimes \delta_y \arrow[r, mapsto] & \delta_{(x,y)} \arrow[r, mapsto] & \delta_{x+y} \end{tikzcd}$$

Indeed, it is sufficient to prove it on Dirac delta distributions as they generate the Mackey-complete space $!E$. □

In general, δ reflects the shape of the functions of its codomain (see Blute et al. (2012) where δ is smooth). In Proposition 5.39, we show that δ is a power series by following the scheme introduced in Ehrhard (2005). First, we focus on the maps $\theta_n : E \to S(E,\mathbb{C})^\times$ that will be the components of the power series δ.

Definition 5.36. Let $\theta_n : E \to S(E,\mathbb{C})^\times$ be defined by induction on n by

$$\theta_0(x) = \delta_0, \qquad \theta_1(x) = \lim_{t\to 0} \frac{\delta_{tx} - \delta_0}{t}, \qquad \forall n \in \mathbb{N},\ \theta_{n+1}(x) = \theta_1(x) * \theta_n(x).$$

Proposition 5.37. For any $n \in \mathbb{N}$, θ_n is a bounded n-monomial from E to $!E$. Besides, for any $x \in E$ and $f = \sum_n f_n \in S(E,\mathbb{C})$, we have $\theta_n(x)f = n!\, f_n(x)$.

Proof. We prove this proposition by induction on $n \in \mathbb{N}$. Let $x \in E$ and $f = \sum_n f_n \in S(E,\mathbb{C})$.

First, θ_0 is constant, $\theta_0(x) = \delta_0$ in $!E$ and $\theta_0(x)f = f(0) = f_0(x)$.

Then, $\theta_1(x)(f) = \lim_{t\to 0} \frac{f(tx)-f(0)}{t} = f_1(x)$. Indeed, by Lemma 5.17, the derivative of $c : z \in \mathbb{C} \mapsto f(zx)$ at 0 is $f_1(x)$. Besides, θ_1 is linear as for $h \in \mathbb{C}$, $\theta_1(x + hy)f = f_1(x + hy) = f_1(x) + hf_1(y)$ by linearity of f_1. Finally, notice that $t \mapsto \delta_{tx}$ is locally Lipschitzian as for any $a > 0$ and $B \subset S(E,\mathbb{C})$ equibounded, the set $\{\frac{f(tx)-f(0)}{t} \mid 0 < t < a, f \in B\} \subset 2B(\{tx \mid 0 < t < a\})$ is bounded. Thus, as proved in Kriegl and Michor (1997, Proposition I.1.7), the net $\left(\frac{\delta_{tx}-\delta_0}{t}\right)_{t\in\mathbb{R}}$ is Mackey-convergent and its limit $\theta_1(x)$ is in the Mackey-complete space $!E$.

Assume that $\theta_n(x)$ is in $!E$ and for any $g = \sum_n g_n$, $\theta_n(x)g = n!\, g_n(x)$. Then, thanks to Proposition 5.35, $\theta_{n+1}(x) = \theta_1(x) * \theta_n(x)$ is in $!E$ and

$$\theta_{n+1}(x)(f) = \theta_1(x)(y \mapsto \theta_n(x)(z \mapsto f(y + z))).$$

By the induction hypothesis,

$$\theta_n(x)(z \mapsto f(y + z)) = n! \sum_{m\geqslant n} \binom{m}{n} \tilde{f}_m(\underbrace{y,\dots,y}_{m-n},\underbrace{x,\dots,x}_{n}),$$

where we denote by $\tilde{f}_m$ the symmetric m-linear bounded map from which the m monomial f_m is constructed. So that,

$$\theta_{n+1}(x)(f) = n! \binom{n+1}{n} \tilde{f}_{n+1}(x,\underbrace{x,\dots,x}_{n}) = (n+1)!\, f_{n+1}(x).$$

□

As in Blute et al. (2012), the differential structure comes from the codereliction. Besides in this setting, this operator extracts the first coefficient of the power series.

Proposition 5.38. The category **Lin** is equipped with a codereliction:

$$\mathrm{coder_E} = \theta_1 : \begin{cases} E \to !E \\ y \mapsto \lim_{t\to 0} \dfrac{\delta(ty) - \delta(0)}{t} \end{cases}$$

Proof. The strength and comonad diagrams of Fiore (2007):

$$
\begin{tikzcd}
E\hat{\otimes}!F \arrow[r, "\mathrm{coder}_E\hat{\otimes}1"] \arrow[dr, "1\hat{\otimes}\epsilon_E"'] & !E\hat{\otimes}!F \arrow[r, "\phi"] & !(E\hat{\otimes}F) \\
& E\hat{\otimes}F \arrow[ur, "\mathrm{coder}_{E\hat{\otimes}F}"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
& !E \arrow[dr, "\epsilon"] & \\
E \arrow[ur, "\mathrm{coder}_E"] \arrow[rr, "1"'] & & E
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "\mathrm{coder}_E"] \arrow[d, "\simeq"] & !E \arrow[r, "\rho"] & !!E \\
E\hat{\otimes}I \arrow[r, "\mathrm{coder}_E\hat{\otimes}v"'] & !E\hat{\otimes}!E \arrow[r, "\mathrm{coder}\hat{\otimes}\rho"'] & !!E\hat{\otimes}!!E \arrow[u, "\nabla"']
\end{tikzcd}
$$

are shown exactly as in Blute et al. (2012) since the actions of the involved natural transformations are defined similarly on the Dirac delta distributions. □

Proposition 5.39. The map δ is a power series in $S(E, S(E,\mathbb{C})^{\times})$:

$$\delta = \sum_{n=0}^{\infty} \frac{\theta_n}{n!}.$$

Proof. In order to show that δ is a power series, we apply Proposition 5.23.

First, notice that δ is bounded from E to $S(E,\mathbb{C})^{\times}$. Indeed, let b be bounded in E, then $\delta(b)$ is equibounded in $S(E,\mathbb{C})^{\times}$, since if B is equibounded in $S(E,\mathbb{C})$, $\delta(b)(B) = B(b)$ is bounded.

Now, let us prove that $\sum_{n=0}^{\infty} \frac{\theta_n}{n!}$ converges pointwise to δ. Let $x \in E$, we need to prove that $\sum_{n=0}^{\infty} \frac{\theta_n(x)}{n!}$ converges to δ_x uniformly on bounded sets of $S(E,\mathbb{C})$. We apply the Cauchy Inequality of Proposition 5.19. Let b be absolutely convex such that $2x \in b$ and $B \in S(E,\mathbb{C})$ be equibounded, then $B(b)$ is bounded in $\mathbb{C}$, i.e. there is M such that $|f(y)| \leqslant M$ for every $f \in B$ and $y \in b$. Thus, for any $f \in B$, $|\frac{\theta_n(x)}{n!}(f)| = \frac{1}{2^n}|f_n(2x)| \leqslant \frac{M}{2^n}$ and the series $\sum_n \frac{\theta_n(x)}{n!}$ converges uniformly on B. Its limit is δ_x as for every $f \in S(E,\mathbb{C})$ and $x \in E$, we have $\delta_x(f) = f(x) = \sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} \frac{\theta_n(x)}{n!}(f)$. From this, we conclude that pointwise, we have $\delta = \sum_{n=0}^{\infty} \theta_n$.

As δ is bounded, Proposition 5.23 implies that the sum uniformly converges on bounded subsets of E. Thus, δ is a power series. □

We just proved that we have a model of Intuitionist Linear Logic and thus, that the cokleisli category $\mathbf{Lin}_!$ is cartesian closed. We want now to show that the category **Series** of Mackey-complete spaces and power series is isomorphic to $\mathbf{Lin}_!$, that is:

Theorem 5.40. For every Mackey-complete space E and F, we have the following bounded isomorphism which is natural in E and F:

$$S(E,F) \simeq \mathcal{L}(!E,F).$$

Proof. Consider $f \in S(E,F)$. Let $\hat{f}_{|\delta(E)} : \delta(E) \to F$ be defined as the linear extension of $\hat{f}(\delta_x) = f(x)$. Since $\hat{f}$ is bounded: $\hat{f}^{-1}_{|\delta(E)}(U) = U_{\{f\},U} \cap \delta(E)$, we can extend it to $!E$ by Mackey-completion, so that $\hat{f} : !E \to F$ is bounded and linear.

Now consider $g \in \mathcal{L}(!E,F)$ and define $\check{g} : E \to F$ by $\check{g}(x) = g(\delta_x) = g \circ \delta$. As g is bounded, we have by Proposition 5.25 that $\check{g} = \sum_k \frac{1}{k!} g(\theta_k)$.

We check that $\hat{\check{g}} = g$, $\check{\hat{f}} = f$, that $g \mapsto \check{g}$ and $f \mapsto \hat{f}$ are both linear and bounded as δ is, and this induces a bounded isomorphism which is natural in E and F. □

This concludes our construction of our denotational model of Linear Logic.

Theorem 5.41. The category **Lin**, equipped with the comonad !, is a quantitative model of intuitionist Linear Logic whose cokleisli category is **Series**, and a differential category.

6. Series is not $*$-autonomous

One of the limits of the approach with bornologies is the extension to $*$-autonomous categories (Barr 1979). Indeed, one could transform this model into a model of (classical) DiLL by considering pairs $(E, E^{\times})$ of Mackey-complete spaces, where $E^{\times}$ denotes the spaces of all bounded linear forms on E. This would be a construction alike the Chu construction.

It is however difficult to have a more intrinsic approach. One could define a notion of b-reflexive space, as a space which equals its bounded bidual $E^{\times\times}$. However, there is no handy Hahn–Banach theorem for bounded linear maps (see Hogbe-Nlend 1970), and one cannot prove that the symmetric monoidal category of b-reflexive Mackey-complete spaces and bounded maps is closed. Let us point out that this problem is not simpler with usual reflexive spaces, as the category of reflexive topological spaces and linear continuous maps is notoriously not closed. For example, if we consider the bidimensional reflexive Hilbert space l^2, the space $\mathcal{B}(l^2)$ of bounded (equivalently continuous) endomorphisms is not reflexive (nor b-reflexive).

7. Conclusion

This paper may be seen as a quantitative adaptation of Blute et al. (2012). It also brings a smooth and general point of view on quantitative semantics. One can try to understand the computing meaning of this structure of power series, as some refinement to quantitative semantics. Indeed, many constructions of the present work relies on the Cauchy formula that power series satisfy. The same phenomenon happens in Girard's Coherent Banach spaces (Girard 1999).

In our model, **Lin** is a concrete example of a differential category (Blute et al. 2006), whilst **Series** should be a concrete example of a cartesian differential category (Blute et al. 2009) and of a cartesian closed differential category (Manzonetto 2012) and so a model of the Differential and the Resource λ-calculi. By displaying the relation between **Lin** and **Series**, we should have a concrete example of the structure exhibited in cartesian differential storage categories (Blute et al. 2015).

The next step now in understanding smooth models of DiLL would be to go towards differentiation in manifolds, which categorical setting has already been studied in Cockett and Cruttwell (2014). One could begin by working on the logic underneath the theory of diffeology (Iglesias-Zemmour 2013).

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References

Abramsky, S., Jagadeesan, R. and Malacaria, P. (2000). Full abstraction for PCF. *Information and Computation* **163** (2) 409–470.

Barr, M. (1979). *∗-Autonomous Categories*, Lecture Notes in Mathematics, volume 752, Springer, Berlin, With an appendix by Po Hsiang Chu.

Blute, R., Cockett, J.R.B. and Seely, R.A.G. (2015). Cartesian differential storage categories. *Theory and Applications of Categories* **30** (18) 620–687.

Blute, R., Panangaden, P. and Seely, R. (1994). Fock space: A model of linear exponential types. *Manuscript, revised version of the MFPS paper Holomorphic Models of Exponential Types in Linear Logic*, 474–512.

Blute, R., Cockett, R. and Seely, R. (2006). Differential categories. *Mathematical Structures in Computer Science* **16** (6) 1049–1083.

Blute, R., Cockett, R. and Seely, R. (2009). Cartesian differential categories. *Theory and Applications of Categories* **22** (23) 622–672.

Blute, R., Ehrhard, T. and Tasson, C. (2012). A convenient differential category. *Cahiers de Topologie et Géométrie Différentielle Catégoriques* **53** (3) 211–232. ISSN 1245-530X.

Bochnak J. and Siciak, J. (1971). Analytic functions in topological vector spaces. *Studia Mathematica* **39** 77–112.

Boman, J. (1967). Differentiability of a function and of its compositions with functions of one variable. *Mathematics Scandinavica* **20** 249–268.

Danos, V. and Ehrhard, T. (2016). Probabilistic coherence spaces as a model of higher-order probabilistic computation. *Information and Computation* **209** (6) 966–991.

Ehrhard, T. (2002). On Köthe sequence spaces and linear logic. *Mathematical Structures in Computer Science* **12** (5) 579–623.

Ehrhard, T. (2005). Finiteness spaces. *Mathematical Structures in Computer Science* **15** (4) 615–646.

Ehrhard, T. (2007). On finiteness spaces and extensional presheaves over the lawvere theory of polynomials. *Journal of Pure and Applied Algebra*, To appear.

Ehrhard, T. (2016). An introduction to differential linear logic: proof-nets, models and antiderivatives. CoRR abs/1606.01642

Ehrhard, T. and Regnier, L. (2003a). The differential lambda-calculus. *Theoretical Computer Science* **309** (1–3) 1–41.

Ehrhard, T. and Regnier, L. (2003b). The differential lambda-calculus. *Theoretical Computer Science* **309** (1–3) 1–41.

Ehrhard, T., Pagani, L. and Tasson, C. (2014). Probabilistic coherence spaces are fully abstract for probabilistic PCF. In: Sewell, P. (ed.), *The 41th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL14,* San Diego, USA. ACM.

Escardó, M. and Heckmann, R. (2001/02). Topologies on spaces of continuous functions. In: *Proceedings of the 16th Summer Conference on General Topology and its Applications (New York)*, volume 26, 545–564.

Fiore, M. (2007). Differential structure in models of multiplicative biadditive intuitionistic linear logic. In: *Proceedings of the Typed Lambda Calculi and Applications, 8th International Conference, TLCA 2007, Paris, France, June 26–28, 2007* 163–177.

Fiore, M., Gambino, N., Hyland, M. and Winskel, G. (2008). The Cartesian closed bicategory of generalised species of structures. *Journal of the London Mathematical Society. Second Series* **77** (1) 203–220.

Frölicher, A. and Kriegl, A. (1988). *Linear Spaces and Differentiation Theory*, Wiley.

Girard, J.-Y. (1986). The system F of variable types, fifteen years later. *Theoretical Computer Science* **45** (2) 159–192.

Girard, J.-Y. (1988). Normal functors, power series and λ-calculus. *Annals of Pure and Applied Logic* **37** (2) 129–177.

Girard, J.-Y. (1999). Coherent Banach spaces: a continuous denotational semantics. *Theoretical Computer Science* **227** (1–2) 275–297. ISSN 0304-3975. Linear logic, I (Tokyo, 1996).

Grothendieck, A. (1953). Sur certains espaces de fonctions holomorphes. I. *Journal für die Reine und Angewandte Mathematik* **192** 35–64.

Hasegawa, R. (2002). Two applications of analytic functors. *Theoretical Computer Science* **272** (1) 113–175.

Hogbe-Nlend, H. (1970). Sur le "problème de hahn-banach" en bornologie. *Comptes Rendus de l'Académie des Sciences Paris Séries A-B* **270** A1320–A1322.

Hogbe-Nlend, H. (1977). *Bornologies and Functional Analysis.* Mathematics Studies, volume 26, North Holland, Amsterdam.

Hyland, M. and Ong, L. (2000). On full abstraction for PCF. *Information and Computation* **163** (2) 285–408.

Iglesias-Zemmour, P. (2013). *Diffeology*, Mathematical Surveys and Monographs, volume 185, American Mathematical Society, Providence, RI.

Jarchow, H. (1981). *Locally convex spaces.* B. G. Teubner, Stuttgart. Mathematical Textbooks.

Kriegl, A. and Michor, P.W. (1997). *The Convenient Setting of Global Analysis*, Mathematical Surveys and Monographs, volume 53, American Mathematical Society, Providence, RI.

Laird, J., Manzonetto, G. and McCusker, G. (2013a). Constructing differential categories and deconstructing categories of games. *Information and Computation* **222** (C) 247–264.

Laird, J., Manzonetto, G., McCusker, G. and Pagani, M. (2013b). Weighted relational models of typed lambda-calculi. In: *Proceedings of the 28th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2013), 25–28 June 2013, New Orleans, USA* 301–310.

Manzonetto, G. (2012). What is a categorical model of the differential and the resource λ-calculi? *Mathematical Structures in Computer Science* **22** (3) 451–520.

Pagani, M., Selinger, P. and Valiron, B. (2014). Applying quantitative semantics to higher-order quantum computing. In: Sewell, P. (ed.), *The 41th Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL14,* San Diego, USA. ACM.

Robertson, A.P. (1970). On bornological products. *Glasgow Mathematical Journal* **11** 37–40.

Robin, J., Cockett, B. and Cruttwell, G.S.H. (2014). Differential structure, tangent structure, and SDG. *Applied Categorical Structures* **22** (2) 331–417.

Schaefer, H.H. (1971). *Topological Vector Spaces*, Graduate Texts in Mathematics, volume 3, Springer-Verlag, New York-Berlin. Third printing corrected.

Ulam, S. (1930). Zur masstheorie in der allgemeinen mengenlehre. *Fundamenta Mathematicae* **16** (1) 140–150.