An explicit formula for the free exponential modality of linear logic

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Abstract. The exponential modality of linear logic associates a commutative comonoid !A to every formula A, in order to duplicate it. Here, we explain how to compute the free commutative comonoid $!A$ in various models of linear logic, using a sequential limit of equalizers. The recipe is simple and elegant, and enables to unify for the first time the miscellaneous constructions of the exponential modality appearing in the literature. It also sheds light on the duplication policy of linear logic. We illustrate its relevance by applying it to two familiar models of linear logic based on coherence spaces, Conway games and we show its limits in finiteness spaces.

1 Introduction

Linear logic is based on the principle that every hypothesis A_i should appear exactly once in a proof of the sequent

$$A_1, \ldots, A_n \quad \vdash \quad B. \tag{1}$$

This logical restriction enables to represent the logic in monoidal categories, along the idea that every formula denotes an object of the category, and every proof of the sequent (1) denotes a morphism

$$A_1 \otimes \cdots \otimes A_n \quad \longrightarrow \quad B$$

where the tensor product is thus seen as a linear kind of conjunction. Note that, for clarity's sake, we use the same notation for a formula A and for its interpretation (or denotation) in the monoidal category.

This linearity policy on proofs seems far too restrictive in order to integrate traditional forms of reasoning, where it is accepted to repeat or to discard an hypothesis in the course of a logical argument. This difficulty is nicely resolved by providing linear logic with an exponential modality, whose task is to strengthen every formula A into a formula $!A$ which may be repeated or discarded. From a semantic point of view, the formula $!A$ is most naturally interpreted as a *comonoid* of the monoidal category. Recall that a comonoid (C, d, u) in a monoidal category $\mathcal{L}$ is defined as an object C equipped with two morphisms

$$d \quad : \quad C \quad \longrightarrow \quad C \otimes C \qquad\qquad\qquad u \quad : \quad C \quad \longrightarrow \quad \mathbf{1}$$

where $\mathbf{1}$ denotes the monoidal unit of the category. The morphism d and u are respectively called the *multiplication* and the *unit* of the comonoid. The two morphisms d

and u are supposed to satisfy *associativity* and *unitality* properties, neatly formulated by requiring that the two diagrams

$$\begin{tikzcd} & C \arrow[dl, bend right, "d"'] \arrow[dr, bend left, "d"] & \\ C \otimes C \arrow[dr, bend right, "C\otimes d"'] & & C \otimes C \arrow[dl, bend left, "d\otimes C"] \\ & C \otimes C \otimes C & \end{tikzcd} \qquad \begin{tikzcd} & C \arrow[dl, bend right, "d"'] \arrow[dr, bend left, "d"] \arrow[dd, "id"] & \\ C \otimes C \arrow[dr, bend right, "u\otimes C"'] & & C \otimes C \arrow[dl, bend left, "C\otimes u"] \\ & C & \end{tikzcd}$$

commute. Note that we draw our diagrams as if the category were *strictly* monoidal, although the usual models of linear logic are only *weakly* monoidal.

The comonoidal structure of the formula $!A$ enables to interpret the *contraction rule* and the *weakening rule* of linear logic

$$\dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{\Gamma, !A, !A, \Delta \vdash B}}{\Gamma, !A, \Delta \vdash B}\ \text{Contraction} \qquad\qquad \dfrac{\dfrac{\begin{matrix}\pi\\ \vdots\end{matrix}}{\Gamma, \Delta \vdash B}}{\Gamma, !A, \Delta \vdash B}\ \text{Weakening}$$

by pre-composing the interpretation of the proof π with the multiplication d in the case of contraction

$$\Gamma \otimes !A \otimes \Delta \xrightarrow{d} \Gamma \otimes !A \otimes !A \otimes \Delta \xrightarrow{\pi} B$$

and with the unit u in the case of weakening

$$\Gamma \otimes !A \otimes \Delta \xrightarrow{u} \Gamma \otimes \Delta \xrightarrow{\pi} B.$$

Besides, linear logic is generally interpreted in a *symmetric* monoidal category, and one requires that the comonoid $!A$ is commutative, this meaning that the following equality holds:

$$A \xrightarrow{\quad d \quad} A \otimes A \xrightarrow{\quad symmetry \quad} A \otimes A \qquad = \qquad A \xrightarrow{\quad d \quad} A \otimes A$$

When linear logic was introduced by Jean-Yves Girard, twenty years ago, it was soon realized by Robert Seely and others that the multiplicative fragment of the logic should be interpreted in a $*$-autonomous category, or at least, a symmetric monoidal closed category $\mathcal{L}$; and that the category should have finite products in order to interpret the additive fragment of the logic, see [9]. A more difficult question was to understand what categorical properties of the exponential modality " ! " were exactly required, in order to define a model of propositional linear logic – that is, including the multiplicative, additive and exponential components of the logic. Nonetheless, Yves Lafont found in his PhD thesis [6] a simple way to define a model of linear logic. Recall that a comonoid morphism between two comonoids (C_1, d_1, u_1) and (C_2, d_2, u_2) is defined as a morphism $f : C_1 \to C_2$ such that the two diagrams

$$\begin{tikzcd} C_1 \arrow[r, "f"] \arrow[d, "d_1"'] & C_2 \arrow[d, "d_2"] \\ C_1 \otimes C_1 \arrow[r, "f\otimes f"] & C_2 \otimes C_2 \end{tikzcd} \qquad \begin{tikzcd} C_1 \arrow[rr, "f"] \arrow[dr, bend right, "u_1"'] & & C_2 \arrow[dl, bend left, "u_2"] \\ & \mathbf{1} & \end{tikzcd}$$

commute. The commutative comonoid $!A$ is freely generated by an object A when there exists a morphism

$$\varepsilon \quad : \quad !A \quad \rightarrow \quad A$$

such that for every morphism

$$f \quad : \quad C \quad \rightarrow \quad A$$

from a commutative comonoid C to the object A, there exists a unique comonoid morphism

$$f^{\dagger} \quad : \quad C \quad \rightarrow \quad !A$$

such that the diagram

$$\begin{tikzcd} & !A \arrow[dd, "\varepsilon"] \\ C \arrow[ur, bend left, "f^{\dagger}"] \arrow[dr, bend right, "f"'] & \\ & A \end{tikzcd} \tag{2}$$

commutes. Lafont noticed that the existence of a free commutative comonoid $!A$ for every object A of a symmetric monoidal closed category $\mathcal{L}$ induces automatically a model of propositional linear logic. But this is not the only way to construct a model of linear logic. A folklore example is the coherence space model, which admits two alternative interpretations of the exponential modality: the original one, formulated by Girard [3] where the coherence space $!A$ is defined as a space of *cliques*, and the free construction, where $!A$ is defined as a space of *multicliques* (cliques with multiplicity) of the original coherence space A.

In this paper, we explain how to construct the free commutative comonoid in the symmetric monoidal categories $\mathcal{L}$ typically encountered in the semantics of linear logic. Our starting point is the well-known formula defining the *symmetric algebra*

$$SA \quad = \quad \bigoplus_{n \in \mathbb{N}} \quad A^{\otimes n} \ / \ \sim_n \tag{3}$$

generated by a vector space A. The formula computes indeed the free commutative monoid associated to the object A in the category of vector spaces over a given field k. Here, the group Σ_n of permutations on $\{1, \ldots, n\}$ acts on the vector space $A^{\otimes n}$, and the vector space $A^{\otimes n} / \sim_n$ of equivalence classes (or orbits) modulo the group action is defined as the coequalizer of the $n!$ symmetries

$$\begin{tikzcd} A^{\otimes n} \arrow[r, shift left, "symmetry"] \arrow[r, phantom, "\cdots"] \arrow[r, shift right, "symmetry"'] & A^{\otimes n} \arrow[r, dotted, "coequalizer"] & A^{\otimes n} / \sim_n \end{tikzcd}$$

in the category of vector spaces. Since a comonoid in the category $\mathcal{L}$ is the same thing as a monoid in the opposite category $\mathcal{L}^{op}$, it is tempting to apply the *dual* formula to (3) in order to define the free commutative comonoid $!A$ generated by an object A in the category $\mathcal{L}$. Although the idea is extremely naive, it is surprisingly close to the truth...

Indeed, one significant aspect of our work is to establish that the equalizer A^n of the $n!$ symmetries

$$A^n \xrightarrow{\text{equalizer}} A^{\otimes n} \underset{symmetry}{\overset{symmetry}{\underset{\cdots}{\rightrightarrows}}} A^{\otimes n} \tag{4}$$

exists in many familiar models of linear logic, and provides there the n-th layer of the free commutative comonoid $!A$ generated by the object A. As we will see in Sections 2 and 3, this principle is nicely illustrated by the equalizer A^n in the category of coherence spaces, which contains the multicliques of cardinality n in the coherence space A ; and by the equalizer A^n in the category of Conway games, which defines the game where Opponent may open up to n copies of the game A, one after the other, in a sequential order.

Of course, the construction of the free exponential modality does not stop here: one still needs to combine the layers A^n together in order to define $!A$ properly. One obvious solution is to apply the dual of formula (3) and to define $!A$ as the infinite cartesian product

$$!A \quad = \quad \mathop{\&}_{n\in\mathbb{N}} \quad A^n. \tag{5}$$

This formula works perfectly well for symmetric monoidal categories $\mathcal{L}$ where the tensor product distributes over the infinite product, in the sense that the canonical morphism

$$X \quad \otimes \quad \left(\mathop{\&}_{n\in\mathbb{N}} \quad A^n \right) \quad \rightarrow \quad \mathop{\&}_{n\in\mathbb{N}} \; (\; X \; \otimes \; A^n \;) \tag{6}$$

is an isomorphism. This algebraic miracle is not so uncommon: it often happens in models of linear logic enriched over commutative monoids – where morphisms (and thus proofs) may be added. A typical illustration is provided by the relational model of linear logic, where the free exponential $!A$ is defined as the set of finite multisets of A, each A^n describing the set of multisets of cardinality n.

On the other hand, the formula (5) is far too optimistic, and does not work in the typical models of linear logic, like coherence spaces, or game semantics. It is quite instructive to apply it to the category of Conway games: the formula defines in that case a game $!A$ where the first move by Opponent selects a component A^n, and thus decides the number n of copies of the game A played subsequently. This departs from the free commutative comonoid $!A$ which we shall describe in Section 3, where Opponent is allowed to open a new copy of the game A at any point of the interaction. So, there remains to understand how the various layers A^n should be combined together, in order to ensure that $!A$ performs this particular copy policy. The temptation is to ask that every layer A^n is "glued" inside the next layer A^{n+1} in order to permit the computation to transit from one layer to the next in the course of interaction.

The most natural way to perform this "glueing" is to introduce the notion of pointed (or affine) object. By pointed object in a monoidal category $\mathcal{L}$, one means a pair (A, u) consisting of an object A and a morphism $u : A \to \mathbf{1}$ to the monoidal unit. So, a pointed object is the same thing as a comonoid, without a comultiplication. It is folklore that the category of pointed objects and pointed morphisms (defined in the expected way) is symmetric monoidal, and *affine* in the sense that its monoidal unit $\mathbf{1}$ is terminal. Once this notion of pointed object introduced, the construction of the free commutative comonoid $!A$ is excessively simple and elegant, and proceeds in three elementary steps.

First step. The object A is transported to the free pointed object $(A_\bullet, u)$ it generates, when this object exists in the monoidal category $\mathcal{L}$. Intuitively, the purpose of the pointed object $A_\bullet$ is to describe one copy of the object A, or none... It is usually quite easy to define: in the case of coherence spaces, the space $A_\bullet = A \,\&\, \mathbf{1}$ is obtained by adding a point to the web of A ; in the case of Conway games, the game $A_\bullet$ is the game A itself, at least when the category is restricted to the Opponent-starting games.

Second step. The object $A^{\leq n}$ is defined as the equalizer $(A_\bullet)^n$ of the diagram

$$A^{\leq n} \xrightarrow{\text{equalizer}} A_\bullet^{\otimes n} \underset{symmetry}{\overset{symmetry}{\rightrightarrows}} A_\bullet^{\otimes n} \tag{7}$$

in the category $\mathcal{L}$. The purpose of $A^{\leq n}$ is to describe all the layers A^k at the same time, for $k \leq n$. Typically, the object $A^{\leq n}$ computed in the category of coherence spaces is the space of all multicliques in A of cardinality less than n.

Third step. It appears that there exists a canonical morphism

$$A^{\leq n} \longleftarrow A^{\leq n+1}$$

induced by the unit u of the pointed object $A_\bullet$. The free commutative comonoid $!A$ generated by A is then defined as the sequential limit of the sequence

$$1 \longleftarrow A^{\leq 1} \longleftarrow A^{\leq 2} \longleftarrow \cdots \longleftarrow A^{\leq n} \longleftarrow A^{\leq n+1} \longleftarrow \cdots$$

The 2-dimensional description of algebraic theories and PROPs recently performed by Melliès and Tabareau [8] ensures then that this recipe in three steps defines the free commutative comonoid $!A$ generated by the object A... as long as the following fundamental property is satisfied by the symmetric monoidal category $\mathcal{L}$: its tensor product should distribute over

1. the equalizer computing the object $A^{\leq n}$,
2. the sequential limit computing the object $!A$.

So, one main purpose of the paper is to establish that this pair of distributivity properties holds for the category of coherence spaces (in Section 2) and for the category of Conway games (in Section 3). In this way, we demonstrate the extraordinary fact that despite their difference in style, the free exponential modalities of coherence spaces and Conway games are based on *exactly* the same limiting process.
In contrast, in the setting of topological vector spaces, both the formula (5) and the free pointed recipe meet their limits. We take the opportunity (in Section 4) to explain the reason why these methods fail in the finiteness space model of differential linear logic recently introduced by Thomas Ehrhard [2].

2 Coherence spaces

In this section, we compute the free exponential modality in the category of coherence spaces defined by Jean-Yves Girard [3]. A *coherence space* $E = (|E|, \frown)$ consists of a set $|E|$ called its *web*, and of a binary reflexive and symmetric relation $\frown$ over E. A *clique* of E is a set X of pairwise coherent elements of the web:

$$\forall e_1, e_2 \in X, \qquad e_1 \frown e_2.$$

We do not recall here the definition of the category **Coh** of coherence spaces (however, the reader will find a brief description of the category in Appendix 1). Just remember that a morphism $R : E \to E'$ in **Coh** is a clique of the coherence space $E \multimap E'$, so in particular, R is a relation on the web $|E| \times |E'|$.

It is easy to see that the tensor product does not distribute over cartesian products: simply observe that the canonical morphism

$$A \otimes (\mathbf{1} \,\&\, \mathbf{1}) \quad \to \quad (A \otimes \mathbf{1}) \,\&\, (A \otimes \mathbf{1})$$

is not an isomorphism. This explains why formula (5) does not work, and why the construction of the free exponential modality requires a sequential limit, along the line described in the introduction.

First step: compute the free affine object. Computing the free pointed (or affine) object on a coherence space E is easy, because the category **Coh** has cartesian products: it is simply given by formula

$$E_\bullet = E \,\&\, 1.$$

It is useful to think of $E \,\&\, 1$ is the space of multicliques of E with *at most* one element: the very first layer of the construction of the free exponential modality. Recall that a multiclique of E is just a multiset on $|E|$ whose underlying set is a clique of E.

Second step: compute the symmetric tensor power $E^{\leq n}$. It is not difficult to see that the equalizer $E^{\leq n}$ of the symmetries

$$(E \,\&\, \mathbf{1})^{\otimes n} \underset{symmetry}{\overset{symmetry}{\rightrightarrows}} (E \,\&\, \mathbf{1})^{\otimes n}$$

is given by the set of multicliques of E with at most n elements, two multicliques being coherent if their union is still a multiclique. As explained in the introduction, one also needs to check that the tensor product distributes over those equalizers. Consider a cone

$$\begin{tikzcd} & Y \arrow[dl, bend right, "R"'] \arrow[dr, bend left, "R'"] & \\ X \otimes (E \,\&\, \mathbf{1})^{\otimes n} \arrow[rr, shift left, "X \otimes symmetry"] \arrow[rr, shift right, "X \otimes symmetry"'] & & X \otimes (E \,\&\, \mathbf{1})^{\otimes n} \end{tikzcd} \tag{8}$$

We can choose the identity among the symmetries. This ensures already that $R = R'$. Next, we show that the morphism R factors uniquely through the morphism

$$X \otimes E^{\leq n} \xrightarrow{X \otimes \ equalizer} X \otimes (E \,\&\, \mathbf{1})^{\otimes n}$$

To that purpose, one defines the relation

$$R^{\leq n} \ : \ Y \ \longrightarrow \ X \otimes E^{\leq n} \qquad \text{by} \qquad y \ R^{\leq n} \ (x, \mu) \quad \text{iff} \quad y \ R \ (x, u)$$

where μ is a multiset of $|E|$ of cardinal less than n, and u is any word of length n whose letters with multiplicity in $|E \mathbin{\&} \mathbf{1}| = |E| \uplus \{*\}$ define the multiset μ. We let the reader check that the definition is correct, that it defines a clique $R^{\leq n}$ of $Y \multimap (X \otimes E^{\leq n})$, and that it is the unique way to factor R through (8).

Third step: compute the sequential limit

$$E^{\leq 0} = \mathbf{1} \longleftarrow E^{\leq 1} = (E \mathbin{\&} \mathbf{1}) \longleftarrow E^{\leq 2} \longleftarrow E^{\leq 3} \cdots$$

whose arrows are (dualized) inclusions from $E^{\leq n}$ into $E^{\leq n+1}$. Again, it is a basic fact that the limit $!E$ of the diagram is given by the set of all finite multicliques, two multicliques being coherent if their union is a multiclique. One also needs to check that the tensor product distributes over the sequential limit. So, consider a cone

$$\begin{tikzcd}
 & & Y \arrow[dll, bend right, "R_0"'] \arrow[dl, bend right, "R_1"] \arrow[d, "R_2"'] \arrow[dr, bend left, "R_3"] & \\
X \otimes 1 & X \otimes (E \mathbin{\&} 1) \arrow[l] & X \otimes E^{\leq 2} \arrow[l] & X \otimes E^{\leq 3} \cdots \arrow[l]
\end{tikzcd}$$

and define the relation

$$R_\infty \;:\; Y \longrightarrow X \otimes !E \qquad \text{by} \qquad y \; R_\infty \; (x, \mu) \quad \text{iff} \quad \exists n, \quad y \; R_n \; (x, u)$$

where μ is a multiset of elements of $|E|$ and the element u of the web of $E^{\leq n}$ is any word of length n whose letters with multiplicity in $|E \mathbin{\&} \mathbf{1}| = |E| \uplus \{*\}$ define the multiset μ. We let the reader check that R_∞ is a clique of $Y \multimap (X \otimes !E)$ and defines the unique way to factor the cone. This concludes the proof that the sequential limit $!E$ defines the free commutative comonoid generated by E in the category **Coh** of coherence spaces.

3 Conway games

In this section, we compute the free exponential modality in the category of Conway games introduced by André Joyal in [4]. One unifying aspect of our approach is that the construction works in exactly the same way as for coherence spaces.

Conway games. A *Conway game* A is an oriented rooted graph (V_A, E_A, λ_A) consisting of (1) a set V_A of vertices called the *positions* of the game; (2) a set $E_A \subset V_A \times V_A$ of edges called the *moves* of the game; (3) a function $\lambda_A : E_A \to \{-1, +1\}$ indicating whether a move is played by Opponent (-1) or by Proponent $(+1)$. We write $\star_A$ for the root of the underlying graph. A Conway game is called *negative* when all the moves starting from its root are played by Opponent.
A *play* $s = m_1 \cdot m_2 \cdot \ldots \cdot m_{k-1} \cdot m_k$ of a Conway game A is a path $s : \star_A \twoheadrightarrow x_k$ starting from the root $\star_A$

$$s : \star_A \xrightarrow{m_1} x_1 \xrightarrow{m_2} \ldots \xrightarrow{m_{k-1}} x_{k-1} \xrightarrow{m_k} x_k$$

Two paths are parallel when they have the same initial and final positions. A play is *alternating* when

$$\forall i \in \{1, \ldots, k-1\}, \qquad \lambda_A(m_{i+1}) = -\lambda_A(m_i).$$

We note Play_A the set of plays of a game A.

Dual. Every Conway game A induces a *dual* game A^* obtained simply by reversing the polarity of moves.

Tensor product. The tensor product $A \otimes B$ of two Conway games A and B is essentially the asynchronous product of the two underlying graphs. More formally, it is defined as:

- $V_{A \otimes B} = V_A \times V_B$,
- its moves are of two kinds :

$$x \otimes y \to \begin{cases} z \otimes y \text{ if } x \to z \text{ in the game } A \\ x \otimes z \text{ if } y \to z \text{ in the game } B, \end{cases}$$

- the polarity of moves is inherited from games A and B.

The unique Conway game 1 with a unique position $\star$ and no move is the neutral element of the tensor product. As usual in game semantics, every play s of the game $A \otimes B$ can be seen as the interleaving of a play $s_{|A}$ of the game A and a play $s_{|B}$ of the game B.

Strategies. A *strategy* σ of a Conway game A is defined as a non empty set of *alternating plays* of even length such that (1) every non empty play starts with an Opponent move; (2) σ is closed by even length prefix; (3) σ is *deterministic*, i.e. for all plays s, and for all moves m, n, n',

$$s \cdot m \cdot n \in \sigma \ \wedge \ s \cdot m \cdot n' \in \sigma \ \Rightarrow \ n = n'.$$

The category of Conway games. The category **Conway** has Conway games as objects, and strategies σ of $A^* \otimes B$ as morphisms $\sigma : A \to B$. The composition is based on the usual "parallel composition plus hiding" technique and the identity is defined by a copycat strategy. The resulting category **Conway** is compact-closed in the sense of [5].

The category **Conway** does not have finite or infinite products. For that reason, we compute the free exponential modality in the full subcategory $\mathbf{Conway}^{-}$ of negative Conway games, which has products. We explain in a later stage how the free construction on the subcategory $\mathbf{Conway}^{-}$ induces a free construction on the whole category.

First step: compute the free affine object. The monoidal unit $\mathbf{1}$ is terminal in the category $\mathbf{Conway}^{-}$. In other words, every negative Conway game may be seen as an affine object in a unique way, by equipping it with the empty strategy $t_A : A \to 1$. In particular, the free affine object $A_\bullet$ is simply A itself.

Second step: compute the symmetric tensor power A^n as the equalizer of the $n!$ symmetries

$$A^{\otimes n} \overset{\overset{symmetry}{\longrightarrow}}{\underset{\underset{symmetry}{\longrightarrow}}{\cdots}} A^{\otimes n}.$$

A simple argument shows that the equalizer $A^n = A^{\leq n}$ is the following Conway game:

– the positions of the game A^n are the finite words $w = x_1 \cdots x_n$ of length n, whose letters are positions x_i of the game A, and such that $x_{i+1} = \star_A$ is the root of A whenever $x_i = \star_A$ is the root of A, for every $1 \leq i < n$. The intuition is that the letter x_k in the position $w = x_1 \cdots x_n$ of the game A^n describes the position of the k-th copy of A, and that the $i+1$-th copy of A cannot be opened by Opponent unless all the i-th copy of A has been already opened.
– its root is the word $\star_{A^n} = \star_A \cdots \star_A$ where the n the positions x_k are at the root $\star_A$ of the game A,
– a move $w \to w'$ is a move played in one copy:

$$w_1 \; x \; w_2 \;\rightarrow\; w_1 \; y \; w_2$$

where $x \to y$ is a move of the game A. Note that the condition on the positions implies that when a new copy of A is opened (that is, when $x = \star_A$) no position in w_1 is at the root, and all the positions in w_2 are at the root.
– the polarities of moves are inherited from the game A in the obvious way.

Note that A^n may be also seen as the subgame of $A^{\otimes n}$ where the $i+1$-th copy of A is always opened after the i-th copy of A.

Third step: compute the sequential limit

$$A^0 = 1 \longleftarrow A^1 = A \longleftarrow A^2 \longleftarrow A^3 \longleftarrow \cdots$$

whose morphisms are the partial copycat strategies $A^n \leftarrow A^{n+1}$ identifying A^n as the subgame of A^{n+1} where only the first n copies of A are played. The limit of this diagram in the category $\mathbf{Conway}^{-}$ is the game A^∞ defined in the same way as $A^{\leq n}$ except that its positions $w = x_1 \cdot x_2 \cdots$ are infinite sequences of positions of A, all of them at the root except for a finite prefix $x_1 \cdots x_k$. We establish in Appendix 2 that A^∞ is indeed the limit of this diagram, and that the tensor product distributes with this limit. From this, we deduce that the sequential limit A^∞ describes the free commutative comonoid in the category $\mathbf{Conway}^{-}$.

It is nice to observe that the free construction extends to the whole category **Conway** of Conway games. A careful study shows that every commutative comonoid in the category of Conway games is in fact a negative game. Moreover, the inclusion functor from $\mathbf{Conway}^{-}$ to **Conway** has a right adjoint, which associates to every Conway game A, the negative Conway game A^{-} obtained by removing all the Proponent moves from the root $\star_A$. By combining these two observations, we obtain that $(A^{-})^\infty$ is the free commutative comonoid generated by A in the category **Conway** of Conway games.

4 An interesting counter-example: Finiteness spaces

In Sections 2 and 3 we have seen how formula (5) can be refined into formula (**??**) which suits the models of Coherent spaces and Conway games. To conclude this paper, we want to give the limits of this approach with the finiteness spaces counter-example.

There are two layers of finiteness spaces. Relational finiteness spaces constitute a refinement of the relational model, built through *bi-orthogonality*. Linear finiteness spaces are linearly topologized spaces [7] built on the relational layer. We explain the failure at both levels. For an introduction to finiteness spaces, we refer the reader to [2] or to Appendix 4.

Relational finiteness spaces. Two subsets u, u' of a countable set $\mathbb{E}$ are said orthogonal, denoted by $u \perp u'$, whenever their intersection $u \cap u'$ is finite. The orthogonal of $\mathcal{G} \subseteq \mathcal{P}(\mathbb{E})$ is then defined by $\mathcal{G}^{\perp} = \{u' \subseteq \mathbb{E} \,|\, \forall u \in \mathcal{G},\ u \perp u'\}$.

A *relational finiteness space* $E = (|E|, \mathcal{F}(E))$ is given by its *web* (the countable set $|E|$) and by a set $\mathcal{F}(E) \subseteq \mathcal{P}(|E|)$ orthogonally closed, i.e. $\mathcal{F}(E)^{\perp\perp} = \mathcal{F}(E)$. The elements of $\mathcal{F}(E)$ (resp. $\mathcal{F}(E)^{\perp}$) are said *finitary* (resp. *antifinitary*). A finitary relation R between two finiteness spaces E_1 and E_2 is a subset of $|E_1| \times |E_2|$ such that

$$\begin{aligned} \forall u \in \mathcal{F}(E_1),\ \ R \cdot u \ &:= \big\{ b \in |E_2| \;\big|\; \exists a \in u,\, (a,b) \in R \big\} \in \mathcal{F}(E_2), \\ \forall v' \in \mathcal{F}(E_2)^{\perp},\ {}^{t}R \cdot v' &:= \big\{ a \in |E_1| \;\big|\; \exists b \in v',\, (a,b) \in R \big\} \in \mathcal{F}(E_1)^{\perp}. \end{aligned}$$

The category **RelFin** of relational finiteness spaces and finitary relations is $*$-autonomous. Thus, it provides a model of linear logic.

The exponential modality of a finiteness space E is the finiteness space $!E$ defined by its web $|!E| = \mathcal{M}_{\text{fin}}(|E|)$ made of finite multisets $\mu : |E| \to \mathbb{N}$ and by its finiteness structure

$$\mathcal{F}(!E) = \{M \in \mathcal{M}_{\text{fin}}(|E|) \mid \pi_E(M) \in \mathcal{F}(E)\},$$

where for every $M \in \mathcal{M}_{\text{fin}}(|E|)$, $\pi_E(M) := \{x \in |E| \mid \exists \mu \in M,\, \mu(x) \neq 0\}$.

The equalizer E^n of the $n!$ symmetries

$$E^n \xrightarrow{\ \ equalizer\ \ } E^{\otimes n} \underset{symmetry}{\overset{symmetry}{\underset{\cdots}{\rightrightarrows}}} E^{\otimes n}$$

exists in **RelFin** and provides the n-th layer of $!E$. Its web $|E^n| = \mathcal{M}^n_{\text{fin}}(|E|)$ is made of the multisets of cardinality n and its finiteness structure is

$$\mathcal{F}(E^n) = \{M_n \subseteq \mathcal{M}^n_{\text{fin}}(|E|) \,|\, \pi_E(M_n) \in \mathcal{F}(E)\}.$$

However, the infinite cartesian product E^{∞} of the layers E^n, i.e. $E^{\infty} = \&_{n\in\mathbb{N}} E^n$, does not compute the exponential modality of **RelFin**. Indeed, although the webs are both equal to $|E^{\infty}| = \mathcal{M}_{\text{fin}}(|E|)$, their finiteness structures are different. As shown by the counter-example in Appendix 3, we have in general $\mathcal{F}(!E) \subsetneq \mathcal{F}(E^{\infty})$ since

$$\mathcal{F}(E^{\infty}) = \left\{ M \in \mathcal{M}_{\text{fin}}(|E|) \;\middle|\; \forall n \in \mathbb{N},\ \begin{matrix} M_n = M \cap \mathcal{M}^n_{\text{fin}}(|E|), \\ \pi_E(M_n) \in \mathcal{F}(E). \end{matrix} \right\}. \tag{9}$$

Besides, the tensor product does not distribute over infinite product, i.e.

$$X \otimes \left(\underset{n\in\mathbb{N}}{\&} E^n \right) \to \underset{n\in\mathbb{N}}{\&} (X \otimes E^n)$$

is not an isomorphism since in general the finiteness structures are different in general

$$\mathcal{F}(X \otimes E^{\infty}) = \{M \subseteq |X| \times \mathcal{M}_{\text{fin}}(|E|) \,|\, \pi_X(M) \in \mathcal{F}(X)\,;\, \pi_E(M) \in \mathcal{F}(E)\}\,,$$

$$\mathcal{F}\left(\underset{n\in\mathbb{N}}{\&} (X \otimes E^n) \right) = \left\{ M \;\middle|\; \forall n \in \mathbb{N},\ \begin{matrix} M_n = M \cap |X| \times \mathcal{M}^n_{\text{fin}}(|E|), \\ \pi_X(M_n) \in \mathcal{F}(X), \pi_E(M_n) \in \mathcal{F}(E). \end{matrix} \right\}. \tag{10}$$

Otherwise, in contrast with coherence spaces, the refined formula (**??**) does not provide more information on $!E$. Roughly speaking, the tensor product $\otimes$ commutes with the finite cartesian product $\&$ in **RelFin**. Therefore the limit E^{∞} coincides with the sequential limit of pointed objects $\&_{n\in\mathbb{N}} E^{\leq n}$.

To better understand this result, we move to the linear finiteness spaces layer. In this setting, the lack of uniformity observed in (9) and (10) can explained in terms of uniform convergence.

Linear finiteness spaces. In the sequel, $\Bbbk$ is an infinite field endowed with the discrete topology i.e. every subset of $\Bbbk$ is open. Every relational finiteness space E generates a vector space, the *linear finiteness space*

$$\Bbbk\langle E\rangle = \left\{x \in \Bbbk^{|E|} \;\middle|\; |x| \in \mathcal{F}(E)\right\},$$

where for any sequence $x \in \Bbbk^{|E|}$, $|x| = \{a \in |E| \mid x_a \neq 0\}$. Using the antifinitary parts of E, this space is endowed by the topology generated by the *fundamental linear neighborhood* of 0

$$V_{J'} = \left\{x \in \Bbbk\langle E\rangle \;\middle|\; |x| \cap J' = \emptyset\right\},\ \forall J' \in \mathcal{F}(E)^{\perp}.$$

More precisely, a subset U of $\Bbbk\langle E\rangle$ is *open* if and only if for each $x \in U$ there is $J'_x \in \mathcal{F}(E)^{\perp}$ such that $x + V_{J'_x} \subseteq U$. Endowed with this topology, $\Bbbk\langle E\rangle$ is a linearly topologized space [7].
The category **LinFin**, with linear finiteness spaces as objects and linear continuous functions as morphisms, is $*$-autonomous and provides a model of linear logic.
Though a linear finiteness space $\Bbbk\langle E\rangle$ is entirely determined by its underlying relational space E, the constructions of linear logic can be described from an algebraic and topological viewpoint *independent* from E. Thus, forgetting the underlying relational layer, linear finiteness spaces will be denoted by X, Y,...

We will now build X^{∞} using its dual since its functional definition is more intuitive. In **LinFin**, the dual space $X^{\perp} = (X \multimap \Bbbk)$ is the linearly topologized space of continuous linear forms. This space is endowed with the topology of uniform convergence on *linearly compact subspaces*, i.e. subspaces $K \subseteq X$ that are closed and have a finitary support $|K| := \cup_{x\in K}|x|$. This *linearly compact open* topology is generated by $\mathcal{W}(K) = \{f \,|\, f(K) = 0\}$ where K ranges over *linearly compact subspaces*[1].
Similarly, $X^{\otimes n} \multimap \Bbbk$ is the space of n-linear forms $\phi : X^{\times n} \to \Bbbk$ which are *hypocontinuous*[2], i.e. $\forall K \subseteq X$ linearly compact, $\exists V$ open s.t. $\forall 1 \leq i \leq n$, $\phi(K^{\times(i-1)} \times V \times K^{\times(n-i)}) = 0$. This space is endowed with the linearly compact open topology generated by $\mathcal{W}(K) = \{\phi \,|\, \phi(K^{\times n}) = 0\}$ where K ranges over *linearly compact subspaces.*
The dual $X^n \multimap \Bbbk$ of the equalizer X^n of the $n!$ symmetries is then the space of symmetric n-linear forms which are hypocontinuous. It is endowed with the linearly compact open topology. It can also be described as the space of *homogeneous polynomials* P of degree n over X, i.e. $P : X \to \Bbbk$ is associated to an hypocontinuous symmetric n-linear form $\phi : X^{\times n} \to \Bbbk$ such that $P(x) = \phi(x, \dots, x)$.

[1] Linear compactness can be defined adapting the intersection property to the linearly topologized setting [7]. We prefer their finitary characterisation which is here more useful.
[2] Hypocontinuity is a notion of continuity in between continuity and separated continuity

Finally, we combine the different layers by taking the infinite cartesian product. The dual $(\&_{n\in\mathbb{N}} X^n) \multimap \Bbbk$ is the space of *polynomials* (finite linear combinations of homogeneous polynomials). In this linear finiteness space, a vector subspace V is open if and only if for every $n \in \mathbb{N}$, there is $K_n \subseteq X$ linearly compact such that $\{P \in X^n \multimap \Bbbk \,|\, P(K_n) = 0\} \subseteq V$.

However, as shown in [1], the dual $!X \multimap \Bbbk$, is the *completion* of the space of polynomials endowed with the linearly compact open topology. In this topology, a vector subspace V is open if and only if there is a linearly compact subspace $K \subseteq X$ such that $\forall n$, $\{P \in X^n \multimap \Bbbk \,|\, P(K) = 0\} \subseteq V$. In other words, the topology is uniform over the different layers. Thanks to the Taylor formula shown in [2], the functions in $!X \multimap \Bbbk$ are *analytic*, i.e. they coincide with the sums of converging series whose n-th term is an homogeneous polynomial of degree n.

Finally, using either combinatorial computations or topological viewpoint, we have seen that $!E$ is different from $\&_n E^n$. Indeed, this latter is related to the local information at each level n though the exponential modality is related to a global information. The second formula (**??**) proposed in this article was a step towards understanding how the different layers combine to construct the exponential modality. Although it is sufficient in coherence spaces and Conway games, it does not in the finiteness spaces where infinitely many layers are needed.

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Appendix 1: Coherence spaces

The coherence relation induces an *incoherence relation* $\asymp$ defined by

$$e_1 \asymp e_2 \quad \iff \quad \neg(e_1 \smile e_2) \quad \text{or} \quad e_1 = e_2.$$

Finite product. The product $E_1 \,\&\, E_2$ of two coherence spaces E_1 and E_2 is defined by $|E_1 \,\&\, E_2| = |E_1| \uplus |E_2|$ and two elements (e, i) and (e', j) of the web are coherent when $i \neq j$ or when $i = j$ and $e \smile e'$.

Tensor product. The tensor product $E_1 \otimes E_2$ of two coherence spaces E_1 and E_2 is defined by $|E_1 \otimes E_2| = |E_1| \times |E_2|$ and two elements (e_1, e_2) and (e_1', e_2') are coherent when

$$e_1 \smile e_1' \quad \text{and} \quad e_2 \smile e_2'.$$

Linear implication. The linear implication $E_1 \multimap E_2$ of two coherence spaces E_1 and E_2 is defined by $|E_1 \multimap E_2| = |E_1| \times |E_2|$ and two elements (e_1, e_2) and (e_1', e_2') of the web are incoherent when

$$e_1 \smile e_1' \quad \text{and} \quad e_2 \asymp e_2'.$$

The category of coherence spaces. The category **Coh** of coherence spaces has coherence spaces as objects and cliques of $E_1 \multimap E_2$ as morphisms from E_1 to E_2. As the web of $E_1 \multimap E_2$ is $|E_1| \times |E_2|$, a morphism can be seen as a relation between $|E_1|$ and $|E_2|$, satisfying additional consistency properties. In particular, identity and composition are defined in the same way as identity and composition in the category of sets and relations. This category is $*$-autonomous and provides a model the multiplicative fragment of linear logic.

Appendix 2: Conway games

Proposition 1. *The game A^∞ is the free exponential of the negative Conway game A.*

Proof. Instead of showing in two steps that A^∞ is the limit of the diagram $\mathcal{A}$ and that the tensor product distributes with this limit, we will directly show that $X \otimes A^\infty$ is the limit of the diagram

$$\begin{tikzcd}
X \otimes \mathcal{A} \;:\; X & X \otimes A \arrow[l, "t_A"'] & X \otimes A^{\otimes 2} \arrow[l, shift right, "X \otimes A \otimes t_A"'] \arrow[l, shift left, "X \otimes t_A \otimes A"] \arrow[loop above, "X \otimes symmetry"] & X \otimes A^{\otimes 3} \cdots \arrow[l, shift right=2, "X \otimes A^{\otimes 2} \otimes t_A"'] \arrow[l, phantom, "\cdots"] \arrow[l, shift left=2, "X \otimes t_A \otimes A^{\otimes 2}"] \arrow[loop above, "X \otimes symmetry"]
\end{tikzcd}$$

Let us define a cone on the diagram $X \otimes \mathcal{A}$ whose origin is $X \otimes A^\infty$. We proceed by defining a interleaving function from plays of A^∞ to plays of A^n, and then by defining a copycat strategy. Given a play $s \cdot m$ of A^∞, we define

$$\langle s \cdot m \rangle = \langle s \rangle \cdot \underline{m}$$

where $\underline{m}$ is the underlying move of m in A. We then define the strategy $\varepsilon_n : A^\infty \to A^{\otimes n}$ by its set of plays

$$\varepsilon_n \quad \stackrel{\text{def}}{=} \quad \{s \in \mathrm{Play}^{even}_{A_1{}^\infty \multimap A_2^{\otimes n}} \mid \forall t \prec^{even} s\ ,\ t_{|A_1{}^\infty} = \langle t_{|A_2^{\otimes n}} \rangle\}.$$

The cone of $X \otimes A^\infty$ on $X \otimes \mathcal{A}$ is then given by the strategies $X \otimes \varepsilon_n$. Note that the scheduling of the opening of moves is enforced by the presence of symmetry in the diagram. This explains why A^∞ is not just the infinite tensor product of A. Let $(B, \alpha : B \to \mathcal{A})$ be a cone on $X \otimes \mathcal{A}$. We have to define a strategy from B to $X \otimes A^\infty$. Let us introduced the strategy $i_n : X \otimes A^{\otimes n} \to X \otimes A^\infty$ which mimics Opponent on X and on the n first copies of A, and which does not answer when Opponent opens the $n+1^{\text{th}}$ copy of A^∞. The strategy $\alpha^{\dagger(n)}$ is defined for all n by the commutative diagram

$$\begin{tikzcd}
B \arrow[r, "\alpha_n"] \arrow[d, "\alpha^{\dagger(n)}"'] & A^{\otimes n} \arrow[dl, "i_n"] \\
A^\infty &
\end{tikzcd}$$

Consider now the diagram

$$\begin{tikzcd}
B \arrow[r, "\alpha_{n+1}"'] \arrow[rr, bend left, "\alpha_n"] \arrow[d, "\alpha^{\dagger(n+1)}"'] & A^{\otimes n+1} \arrow[r, "A^{\otimes n} \otimes t_A"'] \arrow[d, "i_{n+1}"'] & A^{\otimes n} \arrow[d, "i_n"] \\
A^\infty \arrow[r, equal] & A^\infty \arrow[r, equal] & A^\infty
\end{tikzcd}$$

It commutes on all faces except for the right down one which satisfies $i_n \circ (A^n \otimes t_B) \subseteq i_{n+1}$. The clockwise external path is equal $\alpha^{\dagger(n)}$, so we deduce that

$$\alpha^{\dagger(n)} \subseteq \alpha^{\dagger(n+1)}.$$

The comonoidal lifting $\alpha^\dagger$ is then defined by the monotone limit of the $\alpha^{\dagger(n)}$'s:

$$\alpha^\dagger \quad \stackrel{\text{def}}{=} \quad \bigcup_n \alpha^{\dagger(n)}$$

This strategy is a cone morphism because $\varepsilon_n \circ i_n = A^n$, which implies

$$\varepsilon_n \circ \alpha^{\dagger(n)} = \varepsilon_n \circ i_n \circ \alpha_n = \alpha_n.$$

It remains to show that this strategy is unique as a cone morphism. Let β be another cone morphism from B to A^∞. Let us define

$$\beta^{(n)} = i_n \circ \varepsilon_n \circ \beta$$

and remark the two following things

$$\beta^{(n)} = \alpha^{\dagger(n)} \quad \text{and} \quad \beta = \bigcup_n \beta^{(n)}.$$

We deduce that $\alpha^\dagger = \tau$, which concludes the proof.

We clearly have the symmetric property for the diagram $\mathcal{A} \otimes X$. By using this fact for $X = 1$, we obtain that A^∞ is the limit of the diagram $\mathcal{A}$.

Appendix 3: Finiteness spaces

Constructions of linear logic in RelFin.

Orthogonal:

$|E^\perp| = |E| \qquad \mathcal{F}(E^\perp) = \mathcal{F}(E)^\perp$

Products and coproducts:

$|0| = |\top| = \emptyset \quad \mathcal{F}(0) = \mathcal{F}(\top) = \{\emptyset\}$

$|\&_i E_i| = |\oplus_i E_i| = \bigsqcup_i |E_i|$

$$\mathcal{F}(\oplus_i E_i) = \left\{ \begin{array}{l} \uplus_{j\in J} u_j \text{ s.t. } J \text{ finite} \\ \forall j \in J,\, u_j \in \mathcal{F}(E_j) \end{array} \right\}$$

$$\mathcal{F}(\&_i E_i) = \left\{ \begin{array}{l} \uplus_i u_i \text{ s.t.} \\ \forall i \in I,\, u_i \in \mathcal{F}(E_i) \end{array} \right\}$$

Multiplicatives:

$|1| = |\bot| = \{*\} \quad \mathcal{F}(1) = \mathcal{F}(\bot) = \{\emptyset, \{*\}\}$

$|E_1 \parr E_2| = |E_1 \otimes E_2| = |E_1| \times |E_2|$

$$\mathcal{F}(E_1 \parr E_2) = \left\{ \begin{array}{l} R \subseteq |E_1| \times |E_2| \text{ s.t.} \\ \forall u \in \mathcal{F}(E_1)^\perp,\ \ R \cdot u \in \mathcal{F}(E_2) \\ \forall v \in \mathcal{F}(E_2)^\perp,\ {}^tR \cdot v \in \mathcal{F}(E_1) \end{array} \right\}$$

$$\mathcal{F}(E_1 \otimes E_2) = \left\{ \begin{array}{r} w \subseteq |E_1| \times |E_2| \text{ s.t. } \pi_1(w) \in \mathcal{F}(E_1) \\ \pi_2(w) \in \mathcal{F}(E_2) \end{array} \right\}$$

where $\pi_1(w) := \{x_1 \in |E_1| \,|\, \exists x_2 \in |E_2|, (x_1, x_2) \in w\}$
$\pi_2(w) := \{x_2 \in |E_2| \,|\, \exists x_1 \in |E_1|, (x_1, x_2) \in w\}$

Exponentials:

$$|!E| = |?E| = \mathcal{M}_{\text{fin}}(|E|) \ = \ \{\mu : |E| \to \mathbb{N} \,|\, \mu(a) > 0 \text{ for finitely many } a \in |E|\}$$
$$\mathcal{F}(!E) = \{M \subseteq \mathcal{M}_{\text{fin}}(|E|) \ | \ \cup\{|\mu|,\, \mu \in M\} \in \mathcal{F}(E)\}$$
$$\mathcal{F}(?E) = \left\{ M \subseteq \mathcal{M}_{\text{fin}}(|E|) \,|\, \forall u \in \mathcal{F}(E)^\perp, \mathcal{M}_{\text{fin}}(u) \cap M \text{ finite} \right\}$$

Example 1. Let $\mathbb{N}\text{at}$ be the relational finiteness space whose web is the set of integers $\mathbb{N}$ and whose finiteness structure $\mathcal{F}(\mathbb{N}\text{at})$ is the finite powerset $\mathcal{P}_{\text{fin}}(\mathbb{N})$. It is the interpretation of $(!1)^\perp$ in **RelFin**.

The finiteness spaces $!\mathbb{N}\text{at}$ and $\mathbb{N}\text{at}^\infty$ have the same web $\mathcal{M}_{\text{fin}}(\mathbb{N})$ but their finitiness structures are different:

$$\begin{aligned} \mathcal{F}(!\mathbb{N}\text{at}) &= \left\{ M \subseteq \mathcal{M}_{\text{fin}}(\mathbb{N}) \,\middle|\, \pi_{\mathbb{N}\text{at}}(M) \text{ finite} \right\} \\ &= \left\{ M \subseteq \mathcal{M}_{\text{fin}}(\mathbb{N}) \,\middle|\, \exists N \in \mathbb{N}; M \subseteq \mathcal{M}_{\text{fin}}(0, \ldots, N) \right\}, \\ \mathcal{F}(\mathbb{N}\text{at}^\infty) &= \left\{ M \subseteq \mathcal{M}_{\text{fin}}(\mathbb{N}) \,\middle|\, \forall n \in \mathbb{N},\, \pi_{\mathbb{N}\text{at}}(M \cap \mathcal{M}^n_{\text{fin}}(\mathbb{N})) \text{ finite} \right\} \end{aligned}$$

For instance, let us denote μ_n the multiset made of n copies of n and $M = \{\mu_n \,|\, n \in \mathbb{N}\}$. We have $M \in \mathcal{F}(\mathbb{N}\text{at}^\infty)$ but $M \notin \mathcal{F}(!\mathbb{N}\text{at})$.

Similarly, $\mathbb{N}\text{at} \otimes \mathbb{N}\text{at}^\infty$ and $\&_{n\in\mathbb{N}}(\mathbb{N}\text{at} \otimes \mathbb{N}\text{at}^n)$ have the same web $\mathbb{N} \times \mathcal{M}_{\text{fin}}(\mathbb{N})$ but different finiteness structures:

$$\mathcal{F}(\mathbb{N}\text{at} \otimes \mathbb{N}\text{at}^\infty) = \{M \subseteq \mathbb{N} \times \mathcal{M}_{\text{fin}}(\mathbb{N}) \,|\, \exists N; M \subseteq \{0, \ldots, N\} \times \mathcal{M}_{\text{fin}}(0, \ldots, N)\},$$
$$\mathcal{F}\left(\mathop{\&}_{n\in\mathbb{N}} (\mathbb{N}\text{at} \otimes \mathbb{N}\text{at}^n) \right) = \left\{ M \,\middle|\, \forall n \in \mathbb{N},\, \exists N_n \begin{array}{l} M_n = M \cap (\mathbb{N} \times \mathcal{M}^n_{\text{fin}}(\mathbb{N})), \\ M_n \subseteq \{0, \ldots, N_n\} \times \mathcal{M}_{\text{fin}}(0, \ldots, N_n). \end{array} \right\}.$$

For instance, let us denote $M' = \{(n, \mu_n) \,|\, n \in \mathbb{N}\}$. We have $M' \in \mathcal{F}(\mathbb{N}\text{at} \otimes \mathbb{N}\text{at}^\infty)$ but $M' \notin \mathcal{F}(\&_{n\in\mathbb{N}}(\mathbb{N}\text{at} \otimes \mathbb{N}\text{at}^n)$.

Constructions of linear logic in LinFin.

Product and Coproduct. The coproduct $X \oplus Y$ of of linear finiteness spaces X and Y is made of linear combinations of elements of X and Y and is endowed with the product topology. Finite product coincide with finite coproduct. However, the infinite coproduct $\oplus_i X_i$ of the collection of finiteness space X_i is a strict subspace of the infinite product $\&_i X_i$.

Linear implication. The linear implication $X \multimap Y$ of two linear finiteness spaces X and Y is the linearly topologized space of continuous linear functions endowed with the topology of uniform convergence on closed spaces with finitary support. This *linearly compact open* topology is generated by

$$\mathcal{W}(K, V) = \{f \,|\, f(K) \subseteq V\}$$

where K ranges over *linearly compact subspaces* of $\Bbbk\langle X\rangle$, i.e. K is closed and $|K| = \cup_{x\in K}|x|$ is finitary, and V ranges over fundamental neighbourhoods of 0.

Let $\perp = \Bbbk$. The topological dual $X^\perp = X \multimap \perp$ of X is endowed with the compact open topology generated by $\mathcal{W}(K) = \{x' \in X^\perp \,|\, \forall x \in K,\, \langle x', x\rangle = 0\}$ where K ranges over linearly compact subspaces of X.

Inductive tensor product. An n-linear form $\phi : (X_i)^i \to \Bbbk$ over linear finiteness spaces $(X_i)_{i\leq n}$ is *hypocontinuous* if for any (K_i) collection of linearly compact subspaces of X_is (respectively), for any i_0 there exists a fundamental linear neighborhood U_{i_0} such that $\phi(\times X_i) = 0$ where $X_i = K_i$ if $i \neq i_0$ and $X_{i_0} = U_{i_0}$. The inductive tensor product[3] $X \parr Y$ of two linear finiteness spaces X and Y is the space of hypocontinuous bilinear forms over $X^\perp \times Y^\perp$, endowed with the linearly compact open topology generated by $\mathcal{W}(K'_X, K'_Y) = \{\phi \,|\, \phi(K'_X, K'_Y) = 0\}$ where K'_X (resp. K'_Y) range over linearly compact subspaces of X' (resp. Y').

Tensor product. The tensor product $\Bbbk\langle X\rangle \mathbin{\widetilde{\otimes}} \Bbbk\langle Y\rangle$ of two linear finiteness spaces X and Y is the dual of $X^\perp \parr Y^\perp$. It is the topological completion of the algebraic tensor product $\Bbbk\langle X\rangle \otimes \Bbbk\langle Y\rangle$ endowed with the topology induced by $(X^\perp \parr Y^\perp)^\perp$.

Exponential modality. The linear finiteness space $\Bbbk\langle !E^\perp\rangle$ coincides with the completion of the space of polynomial functions over $\Bbbk\langle E\rangle$ endowed with the linearly compact open topology.

Example 2. The linear finiteness space associated with $\mathbb{N}\mathrm{at}$ is the space of finite sequences, denoted $\Bbbk\langle \mathbb{N}\mathrm{at}\rangle = \Bbbk^{(\omega)}$. Its topology is discrete since $\mathbb{N} \in \mathcal{F}(\mathbb{N}\mathrm{at})^\perp$. We can infer that the linearly compact subspaces are the finite dimensional subspaces $\Bbbk^{(\omega)}$.
Hence, $\Bbbk\langle \mathbb{N}\mathrm{at}^n \multimap \Bbbk\rangle$ is the space of every symmetric n-linear forms or equivalently of any homogeneous polynomial $P : \Bbbk^{(\omega)} \to \Bbbk$ of degree n.
The space $\Bbbk\langle \mathbb{N}\mathrm{at}^\infty \multimap \Bbbk\rangle$ is made of every polynomials over $\Bbbk^{(\omega)}$. Its topology is generated by the fundamental system made of subspaces V such that $\forall n \in \mathrm{Nat}$, there exists a finite dimensional $K_n \subseteq \Bbbk^{(\omega)}$ such that $\mathcal{W}_n(K_n) = \left\{P \in \Bbbk\langle \mathbb{N}\mathrm{at}^n\rangle' \;\middle|\; P(K_n)\right\} \subseteq V$.

[3] $X \parr Y$ is an adaptation of the inductive tensor product $\otimes_\varepsilon$ to linearly topologized space.

The space $\Bbbk\langle !\mathbb{N}\mathrm{at} \multimap \Bbbk\rangle$ is made of every formal sums of homogeneous polynomial (possibly infinite). Its topology is generated by $\mathcal{W}(K) = \left\{P \in \Bbbk\langle !E\rangle' \;\middle|\; P(K)\right\}$ where K ranges over finiteness subspaces.