An Introduction to Quantitative Semantics (I)

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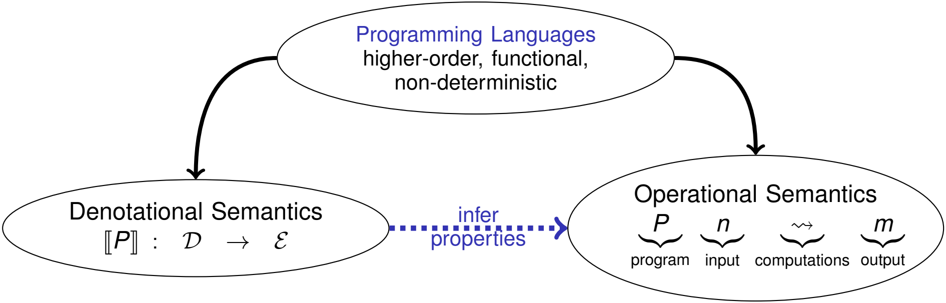
The Big Picture



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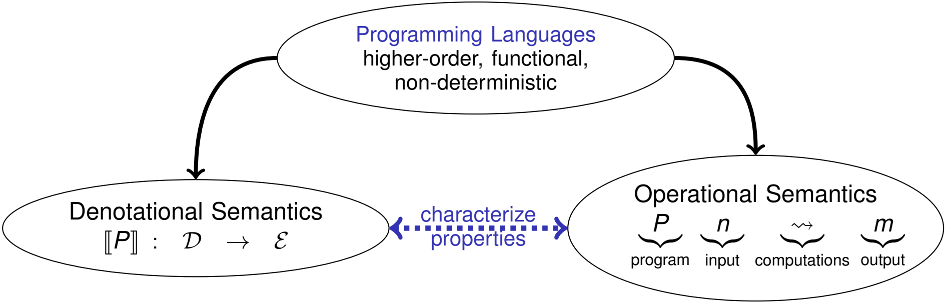

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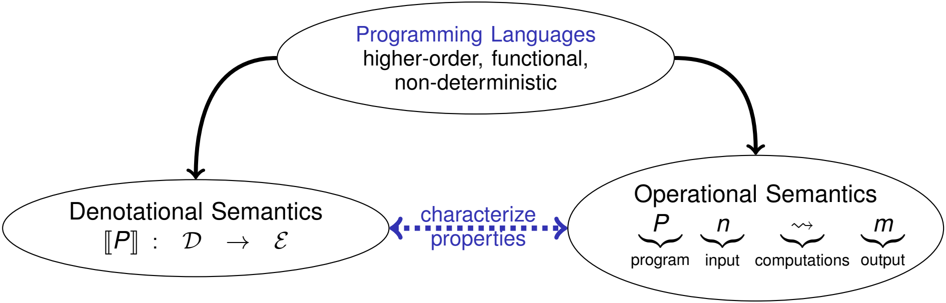

Scott-Strachey's semantics

types : complete partial orders
programs : sup preserving functions
approximants : compact elements

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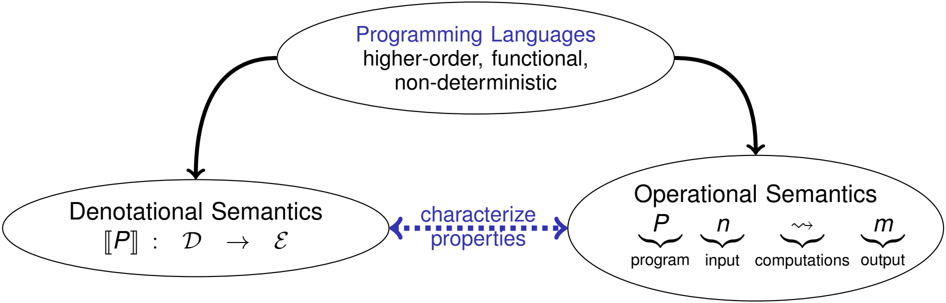

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Qualitative properties

I/O flow
divergence
Morris's observational pre-order

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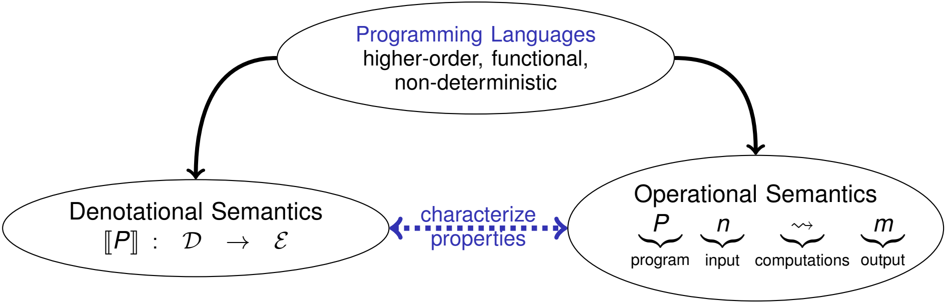

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types : vector spaces
programs : power series
approximants : polynomials

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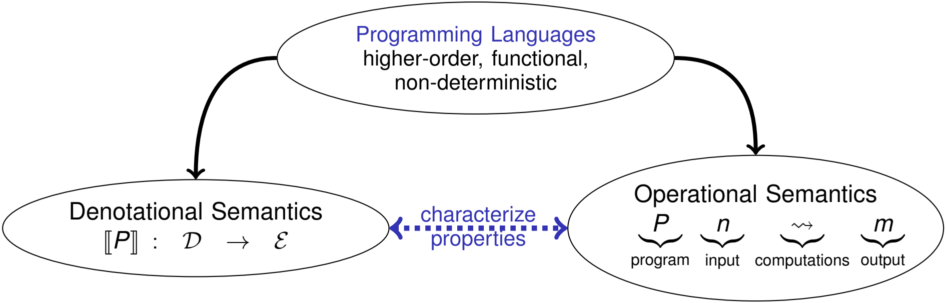

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Quantitative properties

steps to termination
calls to argument
probability to get the output
. . .

Road Map

- Weighted relational semantics
  - the category $\mathfrak{R}^{\Pi}$
  - $\mathfrak{R}^{\Pi}$ is a model of Linear Logic
- A prototypical functional language
  - types and terms
  - operational semantics
  - denotational semantics $\mathfrak{R}^{\Pi}_{!}$
- What can we observe?
  - parametric adequacy
  - some instances

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P.-A. Melliès.
Categorical Semantics of Linear Logic
Panoramas et Synthèses, vol. 27, 2009.

J. Laird and G. Manzonetto and G. McCusker.
Constructing Differential Categories and Deconstructing Categories of Games.
Information and Computation, vol. 222, 2013.

From Relations...

Relational semantics

Objects: sets A, B, C, ...

Morphisms: relations, i.e. subsets in $A \times B$

i.e. matrices in $\text{BOOL}^{A \times B}$

Composition: $A \xrightarrow{\phi} B \xrightarrow{\psi} C$

$$\phi;\psi \triangleq \{(a,c) \mid \exists b, (a,b) \in \phi \text{ and} (b,c) \in \psi\}$$

$$(\phi;\psi)_{a,c} \triangleq \bigvee_{b \in B} \phi_{a,b} \wedge \psi_{b,c}$$

May we generalize to other structures than BOOL?

Ingredients:

- finite commutative multiplication $\wedge$,
- indexed commutative sum $\bigvee$,
- distributivity.

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- finite commutative multiplication $\wedge$,
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A structure $\mathfrak{R} = (R, \cdot, \mathbf{1}, \sum)$ is a complete commutative semi-ring whenever:

- $\cdot$ is a finite commutative multiplication with neutral element $\mathbf{1}$,
- $\sum$ is an indexed commutative sum (empty sum being $\mathbf{0}$),
- distributivity: $\sum_i \kappa_1 \cdot \rho = \left(\sum_i \kappa_1\right) \cdot \rho$.

Example

- $\mathrm{BOOL} = (\{0,1\}, \wedge, 1, \bigvee)$,
- $\mathcal{N} = (\mathbb{N} \cup \{\infty\}, \cdot, 1, \sum)$,
- $\mathcal{T} = (\mathbb{N} \cup \{\infty\}, +, 0, \inf)$,
- $\mathcal{A} = (\mathbb{N} \cup \{+\infty, -\infty\}, +, 0, \sup)$
- $\overline{\mathcal{R}^+} = (\mathbb{R}^+ \cup \{\infty\}, \cdot, 1, \sum)$,
- …

$\mathfrak{R}$-weighted relational semantics

Objects: sets A, B, C, …

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Linear Functions and Duals

We have some key isomorphisms.

$\mathfrak{R}^{A\times B} \simeq \mathrm{Lin}(\mathfrak{R}^A, \mathfrak{R}^B)$

- Take $\phi \in \mathfrak{R}^{A\times B}$, define $\mathrm{Fun}\,(\phi)\,(x)_b \stackrel{\triangle}{=} \sum_a \phi_{a,b} \cdot x_a$
- Take $\phi \in \mathrm{Lin}(\mathfrak{R}^A, \mathfrak{R}^B)$, define $\mathrm{Mat}\,(\phi)_{a,b} \stackrel{\triangle}{=} \phi(e_a)_b$

The interpretation of a proof of $A \vdash B$ is then

a matrix in $\mathfrak{R}^{A\times B}$ as well as a linear function $\mathfrak{R}^A \mapsto \mathfrak{R}^B$

$\mathrm{Lin}(\mathfrak{R}^A, \mathfrak{R}^B) \simeq \mathrm{Lin}((\mathfrak{R}^B)^*, (\mathfrak{R}^A)^*)$, where $(\mathfrak{R}^A)^* \stackrel{\triangle}{=} \mathrm{Lin}(\mathfrak{R}^A, \mathfrak{R})$

is this iso correct? is it necessary for having a model of LL? is it difficult to get on vector spaces, so one introduces topological vector spaces?

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- Take $\phi \in \mathrm{Lin}((\mathfrak{R}^B)^*, (\mathfrak{R}^A)^*)$, define $\phi^* \in \mathrm{Lin}(\mathfrak{R}^A, \mathfrak{R}^B)$ as: $x \mapsto (\phi(e_b^*)(x))_b$

Then, a morphism is the denotation of both

a proof of $A \vdash B$ as well as a proof of $B^\perp \vdash A^\perp$

N.B. proof nets give canonical representatives of this isomorphism

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Denotation of the Multiplicatives

Types: $[\![A^\perp]\!] \overset{\triangle}{=} [\![A]\!]$, i.e. $\mathfrak{R}^{[\![A^\perp]\!]} = (\mathfrak{R}^{[\![A]\!]})^* = \mathfrak{R}^{[\![A]\!]}$

$[\![A \otimes B]\!] = [\![A \parr B]\!] = [\![A \multimap B]\!] \overset{\triangle}{=} [\![A]\!] \times [\![B]\!]$ i.e. $\mathfrak{R}^{[\![A]\!] \times [\![B]\!]} = \mathfrak{R}^{[\![A]\!]} \otimes \mathfrak{R}^{[\![B]\!]}$

$[\![1]\!] = [\![\perp]\!] \overset{\triangle}{=} \{\star\}$, i.e. $\mathfrak{R}^{\{\star\}} = \mathfrak{R}$

Proofs: let $\Gamma = A_1, \dots, A_n$ and $\Delta = B_1, \dots, B_m$, a proof ϕ of $\Gamma \vdash \Delta$ is interpreted by

a matrix in $\mathfrak{R}^{\prod_i [\![A_i]\!] \times \prod_j [\![B_j]\!]}$ as well as a linear function $\bigotimes_i \mathfrak{R}^{[\![A_i]\!]} \mapsto \bigotimes_j \mathfrak{R}^{[\![B_j]\!]}$

$$\left[\!\!\left[\frac{}{A \vdash A}\,\text{ax}\right]\!\!\right] \overset{\triangle}{=} \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \\ \vdots & \vdots & & \ddots \end{pmatrix} \qquad x \mapsto x$$

$$\left[\!\!\left[\frac{\overset{\phi}{\Gamma \vdash B} \qquad \overset{\psi}{B \vdash \Delta}}{\Gamma \vdash \Delta}\,\text{cut}\right]\!\!\right] \overset{\triangle}{=} \phi;\psi \qquad x \mapsto \psi(\phi(x))$$

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$\forall$ bilinear $\phi : \mathfrak{R}^A \times \mathfrak{R}^B \mapsto \mathfrak{R}^C$,
$\exists!$ linear $\phi^\dagger : \mathfrak{R}^A \otimes \mathfrak{R}^B \mapsto \mathfrak{R}^C$, s.t.

$$\phi(x, y) = \phi^\dagger(x \otimes y)$$

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Denotation of the Additives

Types: $[\![0]\!] = [\![\top]\!] = \emptyset$ i.e. $\mathfrak{R}^{\emptyset} = \{\mathbf{0}\}$

$[\![A \mathbin{\&} B]\!] = [\![A \oplus B]\!] = [\![A]\!] \uplus [\![B]\!]$ i.e. $\mathfrak{R}^{[\![A]\!] \uplus [\![B]\!]} = \mathfrak{R}^{[\![A]\!]} \oplus \mathfrak{R}^{[\![B]\!]}$

$[\![\&_{i \in I} A_i]\!] = [\![\bigoplus_{i \in I} A_i]\!] = \bigcup_{i \in I} \{i\} \times [\![A_i]\!]$ i.e. $\mathfrak{R}^{\bigcup_{i \in I} \{i\} \times [\![A_i]\!]} = \bigoplus_{i \in I} \mathfrak{R}^{[\![A]\!]_i}$

Proofs:

$$\left[\!\!\left[\frac{}{\Gamma \vdash \top}\ \mathrm{top} \right]\!\!\right] \triangleq \mathbf{0} \qquad x \mapsto \mathbf{0}$$

$$\left[\!\!\left[\frac{\begin{array}{c}\phi_1 \\ \Gamma \vdash A_1\end{array} \qquad \begin{array}{c}\phi_2 \\ \Gamma \vdash A_2\end{array}}{\Gamma \vdash A_1 \mathbin{\&} A_2}\ \& \right]\!\!\right]_{c,(i,a)} \triangleq (\phi_i)_{c,a} \qquad x \mapsto \phi_1(x) \oplus \phi_2(x)$$

$$\left[\!\!\left[\frac{\begin{array}{c}\phi \\ \Gamma \vdash A_1\end{array}}{\Gamma \vdash A_1 \oplus A_2}\ \oplus_1 \right]\!\!\right]_{c,(i,a)} \triangleq \begin{cases} \phi_{c,a} & \text{if } i = 1, \\ \mathbf{0} & \text{if } i = 2. \end{cases} \qquad x \mapsto \phi(x) \oplus \mathbf{0}$$

$$\left[\!\!\left[\frac{\begin{array}{c}\phi \\ \Gamma \vdash A_2\end{array}}{\Gamma \vdash A_1 \oplus A_2}\ \oplus_2 \right]\!\!\right]_{c,(i,a)} \triangleq \begin{cases} \mathbf{0} & \text{if } i = 1, \\ \phi_{c,a} & \text{if } i = 2. \end{cases} \qquad x \mapsto \mathbf{0} \oplus \phi(x)$$

Denotation of the Additives

Types:

$$[\![0]\!] = [\![\top]\!] = \emptyset \qquad \text{i.e. } \mathfrak{R}^{\emptyset} = \{\mathbf{0}\}$$

$$[\![A \,\&\, B]\!] = [\![A \oplus B]\!] = [\![A]\!] \uplus [\![B]\!] \qquad \text{i.e. } \mathfrak{R}^{[\![A]\!] \uplus [\![B]\!]} = \mathfrak{R}^{[\![A]\!]} \oplus \mathfrak{R}^{[\![B]\!]}$$

$$[\![\&_{i \in I} A_i]\!] = [\![\textstyle\bigoplus_{i \in I} A_i]\!] = \bigcup_{i \in I} \{i\} \times [\![A_i]\!] \qquad \text{i.e. } \mathfrak{R}^{\bigcup_{i \in I} \{i\} \times [\![A_i]\!]} = \textstyle\bigoplus_{i \in I} \mathfrak{R}^{[\![A]\!]_i}$$

$$\forall \text{ object } A, \mathfrak{R}^{A} = \bigoplus_{a \in A} \mathfrak{R}$$

$\mathfrak{R}^{\Pi}$ is the infinite biproduct completion of $\mathfrak{R}$

Proofs:

$$\left[\!\!\left[\frac{}{\Gamma \vdash \top} \text{top} \right]\!\!\right] \triangleq \mathbf{0} \qquad \mapsto \mathbf{0}$$

$$\left[\!\!\left[\frac{\begin{matrix}\phi_1 \\ \Gamma \vdash A_1\end{matrix}}{\Gamma \vdash A_1 \,\&\, A_2} \right]\!\!\right]$$

$$\left[\!\!\left[\frac{\begin{matrix}\phi \\ \Gamma \vdash A_1\end{matrix}}{\Gamma \vdash A_1 \oplus A_2} \oplus_1 \right]\!\!\right]_{c,(i,a)} \triangleq \begin{cases} \phi_{c,a} & \text{if } i = 1, \\ \mathbf{0} & \text{if } i = 2. \end{cases} \qquad x \mapsto \phi(x) \oplus \mathbf{0}$$

$$\left[\!\!\left[\frac{\begin{matrix}\phi \\ \Gamma \vdash A_2\end{matrix}}{\Gamma \vdash A_1 \oplus A_2} \oplus_2 \right]\!\!\right]_{c,(i,a)} \triangleq \begin{cases} \mathbf{0} & \text{if } i = 1, \\ \phi_{c,a} & \text{if } i = 2. \end{cases} \qquad x \mapsto \mathbf{0} \oplus \phi(x)$$

Denotation of the Additives

Types: $[\![0]\!] = [\![\top]\!] = \emptyset$ i.e. $\mathfrak{R}^{\emptyset} = \{\mathbf{0}\}$

$[\![A \,\&\, B]\!] = [\![A \oplus B]\!] = [\![A]\!] \uplus [\![B]\!]$ i.e. $\mathfrak{R}^{[\![A]\!] \uplus [\![B]\!]} = \mathfrak{R}^{[\![A]\!]} \oplus \mathfrak{R}^{[\![B]\!]}$

$[\![\&_{i \in I} A_i]\!] = [\![\bigoplus_{i \in I} A_i]\!] = \bigcup_{i \in I} \{i\} \times [\![A_i]\!]$ i.e. $\mathfrak{R}^{\bigcup_{i \in I}\{i\} \times [\![A_i]\!]} = \bigoplus_{i \in I} \mathfrak{R}^{[\![A]\!]_i}$

Proofs:

$$\left[\!\!\left[\frac{}{\Gamma \vdash \top}\ \mathrm{top} \right]\!\!\right] \triangleq \mathbf{0} \qquad x \mapsto \mathbf{0}$$

$$\left[\!\!\left[\frac{\begin{array}{c}\phi_1\\ \Gamma \vdash A_1\end{array} \qquad \begin{array}{c}\phi_2\\ \Gamma \vdash A_2\end{array}}{\Gamma \vdash A_1 \,\&\, A_2}\ \& \right]\!\!\right]_{c,(i,a)} \triangleq (\phi_i)_{c,a} \qquad x \mapsto \phi_1(x) \oplus \phi_2(x)$$

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$$\left[\!\!\left[\frac{\begin{array}{c}\phi\\ \Gamma \vdash A_2\end{array}}{\Gamma \vdash A_1 \oplus A_2}\ \oplus_2 \right]\!\!\right]_{c,(i,a)} \triangleq \begin{cases} \mathbf{0} & \text{if } i = 1, \\ \phi_{c,a} & \text{if } i = 2. \end{cases} \qquad x \mapsto \mathbf{0} \oplus \phi(x)$$

Denotation of the Exponentials

Types: $[\![!A]\!] = [\![?A]\!] = \mathcal{M}_f([\![A]\!])$ i.e. $\mathfrak{R}^{[\![!A]\!]} = \mathfrak{R}^{\mathcal{M}_f([\![A]\!])} = \mathrm{Sym}(\mathfrak{R}^A) \triangleq \bigoplus_n \mathrm{Sym}^n(\mathfrak{R}^A)$

Proofs:

$$\left[\!\!\left[\frac{\begin{array}{c}\phi\\ \Gamma \vdash \bot\end{array}}{\Gamma \vdash ?A}\ ?w\right]\!\!\right]_{c,m} \triangleq \begin{cases}\phi_{c,*} & \text{if } m = [\,],\\ 0 & \text{otherwise.}\end{cases} \qquad x \mapsto \phi(x) \oplus \mathbf{0}$$

$$\left[\!\!\left[\frac{\begin{array}{c}\phi\\ \Gamma \vdash A\end{array}}{\Gamma \vdash ?A}\ ?d\right]\!\!\right]_{c,m} \triangleq \begin{cases}\phi_{c,a} & \text{if } m = [a],\\ 0 & \text{otherwise.}\end{cases} \qquad x \mapsto \mathbf{0} \oplus \phi(x) \oplus \mathbf{0}$$

$$\left[\!\!\left[\frac{\begin{array}{c}\phi\\ \Gamma \vdash ?A, ?A\end{array}}{\Gamma \vdash ?A}\ c\right]\!\!\right]_{c,m} \triangleq \sum_{\substack{m_1, m_2\\ m_1+m_2=m}} \phi_{c,(m_1,m_2)} \qquad x \mapsto ????$$

$$\left[\!\!\left[\frac{\begin{array}{c}\phi\\ ?\Gamma \vdash A\end{array}}{?\Gamma \vdash !A}\ !\right]\!\!\right]_{m,[a_1,\dots,a_n]} \triangleq \sum_{\substack{(m_1,\dots,m_n)\\ \text{st } \sum_i m_i = m}} \prod_{i=1}^{n} \phi_{m_i,a_i} \qquad x \mapsto \bigoplus_{n=0}^{\infty} \phi(x)^n$$

Denotation of the Exponentials

Types: $[\![!A]\!] = [\![?A]\!] = \mathcal{M}_f([\![A]\!])$

i.e. $\mathfrak{R}^{[\![!A]\!]} = \mathfrak{R}^{\mathcal{M}_f([\![A]\!])}$

$= \mathrm{Sym}(\mathfrak{R}^A) \triangleq \bigoplus_n \mathrm{Sym}^n(\mathfrak{R}^A)$

$\forall$ bilinear and symmetric $\phi : \mathfrak{R}^A \times \mathfrak{R}^A \mapsto \mathfrak{R}^C$,
$\exists!$ linear $\phi^\dagger : \mathrm{Sym}^2(\mathfrak{R}^A) \mapsto \mathfrak{R}^C$, s.t.

$$\phi(x, y) = \phi^\dagger(x \otimes_s y)$$

Proofs:

$$\left[\!\!\left[\frac{\begin{array}{c}\phi\\ \Gamma \vdash ?A, ?A\end{array}}{\Gamma \vdash ?A}\, c\right]\!\!\right]_{c,m} \triangleq \sum_{\substack{m_1, m_2\\ m_1+m_2=m}} \phi_{c,(m_1,m_2)} \qquad x \mapsto ????$$

$$\left[\!\!\left[\frac{\begin{array}{c}\phi\\ ?\Gamma \vdash A\end{array}}{?\Gamma \vdash !A}\, !\right]\!\!\right]_{m,[a_1,\dots,a_n]} \triangleq \sum_{\substack{(m_1,\dots,m_n)\\ \text{st } \sum_i m_i = m}} \prod_{i=1}^{n} \phi_{m_i,a_i} \qquad x \mapsto \bigoplus_{n=0}^{\infty} \phi(x)^n$$

Denotation of the Exponentials

Types: $[\![!A]\!] = [\![?A]\!] = \mathcal{M}_{\mathrm{f}}([\![A]\!])$ i.e. $\mathfrak{R}^{[\![!A]\!]} = \mathfrak{R}^{\mathcal{M}_{\mathrm{f}}([\![A]\!])} = \mathrm{Sym}(\mathfrak{R}^A) \triangleq \bigoplus_n \mathrm{Sym}^n(\mathfrak{R}^A)$

Proofs:

$$\left[\!\!\left[\frac{\begin{array}{c}\phi\\ \Gamma \vdash \bot\end{array}}{\Gamma \vdash ?A}\,?w\right]\!\!\right]_{c,m} \triangleq \begin{cases}\phi_{c,*} & \text{if } m = [\,],\\ 0 & \text{otherwise.}\end{cases} \qquad x \mapsto \phi(x) \oplus \mathbf{0}$$

$$\left[\!\!\left[\frac{\begin{array}{c}\phi\\ \Gamma \vdash A\end{array}}{\Gamma \vdash ?A}\,?d\right]\!\!\right]_{c,m} \triangleq \begin{cases}\phi_{c,a} & \text{if } m = [a],\\ 0 & \text{otherwise.}\end{cases} \qquad x \mapsto \mathbf{0} \oplus \phi(x) \oplus \mathbf{0}$$

$$\left[\!\!\left[\frac{\begin{array}{c}\phi\\ \Gamma \vdash ?A, ?A\end{array}}{\Gamma \vdash ?A}\,c\right]\!\!\right]_{c,m} \triangleq \sum_{\substack{m_1, m_2\\ m_1 + m_2 = m}} \phi_{c,(m_1,m_2)} \qquad x \mapsto ????$$

$$\left[\!\!\left[\frac{\begin{array}{c}\phi\\ ?\Gamma \vdash A\end{array}}{?\Gamma \vdash !A}\,!\right]\!\!\right]_{m,[a_1,\ldots,a_n]} \triangleq \sum_{\substack{(m_1,\ldots,m_n)\\ \text{st } \sum_i m_i = m}} \prod_{i=1}^{n} \phi_{m_i,a_i} \qquad x \mapsto \bigoplus_{n=0}^{\infty} \phi(x)^n$$

Road Map

- Weighted relational semantics
  - the category $\mathfrak{R}^{\Pi}$
  - $\mathfrak{R}^{\Pi}$ is a model of Linear Logic
- A prototypical functional language
  - types and terms
  - operational semantics
  - denotational semantics $\mathfrak{R}^{\Pi}_{!}$
- What can we observe?
  - parametric adequacy
  - some instances

P.-A. Melliès.
Categorical Semantics of Linear Logic
Panoramas et Synthèses, vol. 27, 2009.

J. Laird and G. Manzonetto and G. McCusker.
Constructing Differential Categories and Deconstructing Categories of Games.
Information and Computation, vol. 222, 2013.

Road Map

- Weighted relational semantics
  - the category $\mathfrak{R}^{\Pi}$
  - $\mathfrak{R}^{\Pi}$ is a model of Linear Logic
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  - types and terms
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  - denotational semantics $\mathfrak{R}^{\Pi}_{!}$
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  - parametric adequacy
  - some instances

G. Plotkin
LCF considered as a programming language
Theoretical Computer Science, 1975.

J.-Y. Girard.
Normal Functors, power series and lambda-calculus
Annals of Pure and Applied Logic, 1988.

Types: $\texttt{Bool} \mid \texttt{Int} \mid A \Rightarrow B$

Terms:

$$x \mid \lambda x^A.P \mid (P)\,Q \mid \mathtt{fix}(P) \qquad \lambda\text{-calculus with recursion}$$

$$\mid \underline{n} \mid P{-}1 \mid P{+}1 \mid \mathtt{iszero}(P) \qquad \text{arithmetic}$$

$$\mid \mathtt{tt} \mid \mathtt{ff} \mid \mathtt{if}(N, P, Q) \qquad \text{booleans}$$

- PCF^c extends PCF by adding a non-deterministic primitive `Coin`.
- $PCF^{\mathfrak{R}}$ extends PCF^c by adding a scalar multiplication κM, for every $\kappa \in \mathfrak{R}$.

Example

$$\mathtt{or} \triangleq \lambda x^{\mathtt{Bool}}.\lambda y^{\mathtt{Bool}}.\mathtt{if}(x, \mathtt{tt}, y)$$

$$\Omega \triangleq \mathtt{fix}(\lambda x^A.x)$$

$$\mathtt{letrec}\, f^{A\Rightarrow B}\, x^A = M \triangleq \mathtt{fix}(\lambda f^{A\Rightarrow B}.\lambda x^A.M)$$

$$\mathtt{double} \triangleq \mathtt{letrec}\, f^{\mathtt{Int}\Rightarrow\mathtt{Int}}\, x^{\mathtt{Int}} = \mathtt{if}(\mathtt{iszero}(x), \underline{0}, ((f)\,(x{-}1)){+}1{+}1)$$

$$\mathtt{waittt} \triangleq \mathtt{letrec}\, f^{\mathtt{Bool}\Rightarrow\mathtt{Bool}}\, x^{\mathtt{Bool}} = \mathtt{if}(x, \mathtt{tt}, (f)\,x)$$

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{}{\Gamma \vdash \underline{n} : \mathtt{Int}} \qquad \frac{}{\Gamma \vdash \mathtt{tt} : \mathtt{Bool}} \qquad \frac{}{\Gamma \vdash \mathtt{ff} : \mathtt{Bool}}$$

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^A.M : A \Rightarrow B} \qquad \frac{\Gamma \vdash M : A \Rightarrow B \quad \Gamma \vdash N : A}{\Gamma \vdash (M)\, N : B} \qquad \frac{\Gamma \vdash M : A \Rightarrow A}{\Gamma \vdash \mathtt{fix}(M) : A}$$

$$\frac{\Gamma \vdash P : \mathtt{Int}}{\Gamma \vdash P{+}1 : \mathtt{Int}} \qquad \frac{\Gamma \vdash P : \mathtt{Int}}{\Gamma \vdash P{-}1 : \mathtt{Int}} \qquad \frac{\Gamma \vdash P : \mathtt{Int}}{\Gamma \vdash \mathtt{iszero}(P) : \mathtt{Bool}}$$

$$\frac{\Gamma \vdash M : \mathtt{Bool} \qquad \Gamma \vdash N : A \qquad \Gamma \vdash P : A}{\Gamma \vdash \mathtt{if}(M, N, P) : A}$$

$$\frac{}{\Gamma \vdash \mathtt{Coin} : \mathtt{Bool}} \qquad \frac{\Gamma \vdash M : A}{\Gamma \vdash \kappa M : A}$$

Operational Semantics – Small Step

Example ($\mathtt{or} \triangleq \lambda x^{\mathtt{Bool}}.\lambda y^{\mathtt{Bool}}.\mathtt{if}(x, \mathtt{tt}, y)$)

$$((\mathtt{or})\mathtt{tt})\,\mathtt{ff} \xrightarrow{1} (\lambda^{\mathtt{Bool}} y.\,\mathtt{if}(\mathtt{tt}, \mathtt{tt}, y))\,\mathtt{ff}$$
$$\xrightarrow{1} \mathtt{If}(\mathtt{tt}, \mathtt{tt}, \mathtt{ff})$$
$$\xrightarrow{1} \mathtt{tt}$$

Redex-to-contractum rules

$$\left(\lambda x^{A}.M\right) N \xrightarrow{1} M\{N/x\} \qquad \mathtt{fix}(M) \xrightarrow{1} (M)\,\mathtt{fix}(M)$$

$$\mathtt{iszero}(\underline{0}) \xrightarrow{1} \mathtt{tt} \qquad \mathtt{iszero}(\underline{n+1}) \xrightarrow{1} \mathtt{ff} \qquad \underline{n}{+}1 \xrightarrow{1} \underline{n+1} \qquad \underline{n+1}{-}1 \xrightarrow{1} \underline{n}$$

$$\mathtt{if}(\mathtt{tt}, M, N) \xrightarrow{1} M \qquad \mathtt{if}(\mathtt{ff}, M, N) \xrightarrow{1} N \qquad \mathtt{Coin} \xrightarrow{1} \mathtt{tt} \qquad \mathtt{Coin} \xrightarrow{1} \mathtt{ff} \qquad \kappa M \xrightarrow{\kappa} M$$

Contextual rules supposing $M \xrightarrow{\kappa} N$

$$\text{if } M \text{ not a } \lambda,\ (M)\,P \xrightarrow{\kappa} (N)\,P \qquad \mathtt{iszero}(M) \xrightarrow{\kappa} \mathtt{iszero}(N)$$

$$M{+}1 \xrightarrow{\kappa} N{+}1 \qquad M{-}1 \xrightarrow{\kappa} N{-}1 \qquad \mathtt{if}(M, P, P') \xrightarrow{1} \mathtt{if}(N, P, P')$$

Operational Semantics – Small Step

Example ($\mathtt{or} \triangleq \lambda x^{\mathtt{Bool}}.\lambda y^{\mathtt{Bool}}.\mathtt{if}(x,\mathtt{tt},y)$)

$$((\mathtt{or})\mathtt{tt})\,\mathtt{ff} \xrightarrow{1} (\lambda^{\mathtt{Bool}} y.\,\mathtt{if}(\mathtt{tt},\mathtt{tt},y))\,\mathtt{ff}$$

$$\xrightarrow{1} \mathtt{If}(\mathtt{tt},\mathtt{tt},\mathtt{ff})$$

$$\xrightarrow{1} \mathtt{tt}$$

Redex-to-contractum rules

$$\left(\lambda x^A.M\right) N \xrightarrow{1} M\{N/x\} \qquad \mathtt{fix}(M) \xrightarrow{1} (M)\,\mathtt{fix}(M)$$

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$$\mathtt{if}(\mathtt{tt},M,N) \xrightarrow{1} M \qquad \mathtt{if}(\mathtt{ff},M,N) \xrightarrow{1} N \qquad \mathtt{Coin} \xrightarrow{1} \mathtt{tt} \qquad \mathtt{Coin} \xrightarrow{1} \mathtt{ff} \qquad \kappa M \xrightarrow{\kappa} M$$

Contextual rules — supposing $M \xrightarrow{\kappa} N$

if M not a λ, $(M)\,P \xrightarrow{\kappa} (N)\,P \qquad \mathtt{iszero}(M) \xrightarrow{\kappa} \mathtt{iszero}(N)$

$$M{+}1 \xrightarrow{\kappa} N{+}1 \qquad M{-}1 \xrightarrow{\kappa} N{-}1 \qquad \mathtt{if}(M,P,P') \xrightarrow{1} \mathtt{if}(N,P,P')$$

Example ($\mathtt{or} \triangleq \lambda x^{\mathtt{Bool}}.\lambda y^{\mathtt{Bool}}.\mathtt{if}(x, \mathtt{tt}, y)$)

$$((\mathtt{or})\mathtt{tt})\,\mathtt{ff} \xrightarrow{1} (\lambda^{\mathtt{Bool}} y.\,\mathtt{if}(\mathtt{tt}, \mathtt{tt}, y))\,\mathtt{ff}$$
$$\xrightarrow{1} \mathtt{If}(\mathtt{tt}, \mathtt{tt}, \mathtt{ff})$$
$$\xrightarrow{1} \mathtt{tt}$$

Redex-to-contractum rules

$$(\lambda x^A.M)\,N \xrightarrow{1} M\{N/x\} \qquad \mathtt{fix}(M) \xrightarrow{1} (M)\,\mathtt{fix}(M)$$

$$\mathtt{iszero}(\underline{0}) \xrightarrow{1} \mathtt{tt} \qquad \mathtt{iszero}(\underline{n+1}) \xrightarrow{1} \mathtt{ff} \qquad \underline{n}{+}1 \xrightarrow{1} \underline{n+1} \qquad \underline{n+1}{-}1 \xrightarrow{1} \underline{n}$$

$$\mathtt{if}(\mathtt{tt}, M, N) \xrightarrow{1} M \qquad \mathtt{if}(\mathtt{ff}, M, N) \xrightarrow{1} N \qquad \mathtt{Coin} \xrightarrow{1} \mathtt{tt} \qquad \mathtt{Coin} \xrightarrow{1} \mathtt{ff} \qquad \kappa M \xrightarrow{\kappa} M$$

Contextual rules supposing $M \xrightarrow{\kappa} N$

$$\text{if } M \text{ not a } \lambda,\ (M)\,P \xrightarrow{\kappa} (N)\,P \qquad \mathtt{iszero}(M) \xrightarrow{\kappa} \mathtt{iszero}(N)$$

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Operational Semantics – Small Step

Example ($\mathtt{or} \triangleq \lambda x^{\mathtt{Bool}}.\lambda y^{\mathtt{Bool}}.\mathtt{if}(x, \mathtt{tt}, y)$)

$$((\mathtt{or})\mathtt{tt})\,\mathtt{ff} \xrightarrow{1} (\lambda^{\mathtt{Bool}} y.\,\mathtt{if}(\mathtt{tt}, \mathtt{tt}, y))\,\mathtt{ff}$$

$$\xrightarrow{1} \mathtt{If}(\mathtt{tt}, \mathtt{tt}, \mathtt{ff})$$

$$\xrightarrow{1} \mathtt{tt}$$

Redex-to-contractum rules

$$(\lambda x^A.M)\,N \xrightarrow{1} M\{N/x\} \qquad \mathtt{fix}(M) \xrightarrow{1} (M)\,\mathtt{fix}(M)$$

$$\mathtt{iszero}(\underline{0}) \xrightarrow{1} \mathtt{tt} \qquad \mathtt{iszero}(\underline{n+1}) \xrightarrow{1} \mathtt{ff} \qquad \underline{n}{+}1 \xrightarrow{1} \underline{n+1} \qquad \underline{n+1}{-}1 \xrightarrow{1} \underline{n}$$

$$\mathtt{if}(\mathtt{tt}, M, N) \xrightarrow{1} M \qquad \mathtt{if}(\mathtt{ff}, M, N) \xrightarrow{1} N \qquad \mathtt{Coin} \xrightarrow{1} \mathtt{tt} \qquad \mathtt{Coin} \xrightarrow{1} \mathtt{ff} \qquad \kappa M \xrightarrow{\kappa} M$$

Contextual rules — supposing $M \xrightarrow{\kappa} N$

$$\text{if } M \text{ not a } \lambda,\ (M)\,P \xrightarrow{\kappa} (N)\,P \qquad \mathtt{iszero}(M) \xrightarrow{\kappa} \mathtt{iszero}(N)$$

$$M{+}1 \xrightarrow{\kappa} N{+}1 \qquad M{-}1 \xrightarrow{\kappa} N{-}1 \qquad \mathtt{if}(M, P, P') \xrightarrow{1} \mathtt{if}(N, P, P')$$

Operational Semantics – Small Step

Example ($\mathtt{or} \triangleq \lambda x^{\mathtt{Bool}}.\lambda y^{\mathtt{Bool}}.\mathtt{if}(x, \mathtt{tt}, y)$)

$$((\mathtt{or})\mathtt{tt})\,\mathtt{ff} \xrightarrow{1} (\lambda^{\mathtt{Bool}} y.\,\mathtt{if}(\mathtt{tt}, \mathtt{tt}, y))\,\mathtt{ff}$$

$$\xrightarrow{1} \mathtt{If}(\mathtt{tt}, \mathtt{tt}, \mathtt{ff})$$

$$\xrightarrow{1} \mathtt{tt}$$

Redex-to-contractum rules

$$\left(\lambda x^A.M\right) N \xrightarrow{1} M\{N/x\} \qquad \mathtt{fix}(M) \xrightarrow{1} (M)\,\mathtt{fix}(M)$$

$$\mathrm{iszero}(\underline{0}) \xrightarrow{1} \mathtt{tt} \qquad \mathrm{iszero}(\underline{n+1}) \xrightarrow{1} \mathtt{ff} \qquad \underline{n}{+}1 \xrightarrow{1} \underline{n+1} \qquad \underline{n+1}{-}1 \xrightarrow{1} \underline{n}$$

$$\mathtt{if}(\mathtt{tt}, M, N) \xrightarrow{1} M \qquad \mathtt{if}(\mathtt{ff}, M, N) \xrightarrow{1} N \qquad \mathtt{Coin} \xrightarrow{1} \mathtt{tt} \qquad \mathtt{Coin} \xrightarrow{1} \mathtt{ff} \qquad \kappa M \xrightarrow{\kappa} M$$

Contextual rules

supposing $M \xrightarrow{\kappa} N$

if M not a λ, $(M)\,P \xrightarrow{\kappa} (N)\,P$ $\qquad \mathrm{iszero}(M) \xrightarrow{\kappa} \mathrm{iszero}(N)$

$$M{+}1 \xrightarrow{\kappa} N{+}1 \qquad M{-}1 \xrightarrow{\kappa} N{-}1 \qquad \mathtt{if}(M, P, P') \xrightarrow{1} \mathtt{if}(N, P, P')$$

Operational Semantics – Small Step

Example ($\mathtt{or} \triangleq \lambda x^{\mathtt{Bool}}.\lambda y^{\mathtt{Bool}}.\mathtt{if}(x, \mathtt{tt}, y)$)

$$((\mathtt{or})\mathtt{tt})\,\mathtt{ff} \xrightarrow{1} (\lambda^{\mathtt{Bool}} y.\,\mathtt{if}(\mathtt{tt}, \mathtt{tt}, y))\,\mathtt{ff}$$

$$\xrightarrow{1} \mathtt{If}(\mathtt{tt}, \mathtt{tt}, \mathtt{ff})$$

$$\xrightarrow{1} \mathtt{tt}$$

Redex-to-contractum rules

$$\left(\lambda x^{A}.M\right) N \xrightarrow{1} M\{N/x\} \qquad \mathtt{fix}(M) \xrightarrow{1} (M)\,\mathtt{fix}(M)$$

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$$\mathtt{if}(\mathtt{tt}, M, N) \xrightarrow{1} M \qquad \mathtt{if}(\mathtt{ff}, M, N) \xrightarrow{1} N \qquad \mathtt{Coin} \xrightarrow{1} \mathtt{tt} \qquad \mathtt{Coin} \xrightarrow{1} \mathtt{ff} \qquad \kappa M \xrightarrow{\kappa} M$$

Contextual rules — supposing $M \xrightarrow{\kappa} N$

$$\text{if } M \text{ not a } \lambda,\ (M)\,P \xrightarrow{\kappa} (N)\,P \qquad \mathtt{iszero}(M) \xrightarrow{\kappa} \mathtt{iszero}(N)$$

$$M{+}1 \xrightarrow{\kappa} N{+}1 \qquad M{-}1 \xrightarrow{\kappa} N{-}1 \qquad \mathtt{if}(M, P, P') \xrightarrow{1} \mathtt{if}(N, P, P')$$

Example ($\mathtt{or} \stackrel{\triangle}{=} \lambda x^{\mathtt{Bool}}.\lambda y^{\mathtt{Bool}}.\mathtt{if}(x, \mathtt{tt}, y)$)

$$((\mathtt{or})\mathtt{tt})\,\mathtt{ff} \xrightarrow{1} (\lambda^{\mathtt{Bool}} y.\,\mathtt{if}(\mathtt{tt}, \mathtt{tt}, y))\,\mathtt{ff}$$

$$\xrightarrow{1} \mathtt{If}(\mathtt{tt}, \mathtt{tt}, \mathtt{ff})$$

$$\xrightarrow{1} \mathtt{tt}$$

Redex-to-contractum rules

$$\left(\lambda x^A.M\right) N \xrightarrow{1} M\{N/x\}$$

$\mathtt{iszero}(\underline{0}) \xrightarrow{1} \mathtt{tt}$ ... $\xrightarrow{1} \underline{n}$

$\mathtt{if}(\mathtt{tt}, M, N) \xrightarrow{1} M$ ... $\xrightarrow{1} \mathtt{tt}$ ... $\mathtt{n} \xrightarrow{1} \mathtt{ff}$ $\quad \kappa M \xrightarrow{\kappa} M$

$\mathtt{if}(P, M, P') \xrightarrow{\kappa} \mathtt{if}(P, N, P')$

$\mathtt{if}(P, P', M) \xrightarrow{\kappa} \mathtt{if}(P, P', N)$

are not allowed

$(P)\,M \xrightarrow{\kappa} (P)\,N$

is not allowed

Contextual rules

supposing $M \xrightarrow{\kappa} N$

if M not a λ, $(M)\,P \xrightarrow{\kappa} (N)\,P$ $\quad \mathtt{iszero}(M) \xrightarrow{\kappa} \mathtt{iszero}(N)$

$M{+}1 \xrightarrow{\kappa} N{+}1$ $\quad M{-}1 \xrightarrow{\kappa} N{-}1$ $\quad \mathtt{if}(M, P, P') \xrightarrow{1} \mathtt{if}(N, P, P')$

Example ($\Omega \triangleq \mathtt{fix}(\lambda x^A.x)$)

$$\Omega \xrightarrow{1} \left(\lambda x^A.x\right) \mathtt{fix}(\lambda x^A.x)$$
$$\xrightarrow{1} \Omega$$

Example ($\frac{1}{2}\mathtt{Coin}$)

$$\frac{1}{2}\mathtt{Coin} \xrightarrow{\frac{1}{2}} \mathtt{Coin}$$
$$\bullet \xrightarrow{1} \mathtt{tt}$$
$$\bullet \xrightarrow{1} \mathtt{ff}$$

Example (double $\stackrel{\triangle}{=}$ $\texttt{letrec}\, f^{\texttt{Int}\Rightarrow\texttt{Int}}\, x^{\texttt{Int}} = \texttt{if}(\texttt{iszero}(x), \underline{0}, ((f)\,(x{-}1)){+}1{+}1)$)

$$
\begin{aligned}
(\text{double})\,\underline{1} &\xrightarrow{\mathbf{1}} ((\lambda f.\lambda x.\,\texttt{if}(\texttt{iszero}(x), \underline{0}, ((f)\,(x{-}1)){+}1{+}1))\text{double})\underline{1} \\
&\xrightarrow{\mathbf{1}} (\lambda x.\,\texttt{if}(\texttt{iszero}(x), \underline{0}, ((\text{double})\,(x{-}1)){+}1{+}1))\underline{1} \\
&\xrightarrow{\mathbf{1}} \texttt{if}(\texttt{iszero}(\underline{1}), \underline{0}, ((\text{double})\,(\underline{1}{-}1)){+}1{+}1) \\
&\xrightarrow{\mathbf{1}} \texttt{if}(\texttt{ff}, \underline{0}, ((\text{double})\,(\underline{1}{-}1)){+}1{+}1) \\
&\xrightarrow{\mathbf{1}} ((\text{double})\,(\underline{1}{-}1)){+}1{+}1 \\
&\xrightarrow{\mathbf{1}} ((\lambda f.\lambda x.\,\texttt{if}(\texttt{iszero}(x), \underline{0}, ((f)\,(x{-}1)){+}1{+}1))\text{double}))\,(\underline{1}{-}1){+}1{+}1 \\
&\xrightarrow{\mathbf{1}} ((\lambda x.\,\texttt{if}(\texttt{iszero}(x), \underline{0}, ((\text{double})\,(x{-}1)){+}1{+}1)))\,(\underline{1}{-}1)){+}1{+}1 \\
&\xrightarrow{\mathbf{1}} (\,\texttt{if}(\texttt{iszero}((\underline{1}{-}1)), \underline{0}, ((\text{double})\,((\underline{1}{-}1){-}1)){+}1{+}1))){+}1{+}1 \\
&\xrightarrow{\mathbf{1}} (\texttt{if}(\texttt{iszero}(\underline{0}), \underline{0}, ((\text{double})\,((\underline{1}{-}1){-}1)){+}1{+}1))){+}1{+}1 \\
&\xrightarrow{\mathbf{1}} (\texttt{if}(\texttt{tt}, \underline{0}, ((\text{double})\,((\underline{1}{-}1){-}1)){+}1{+}1))){+}1{+}1 \xrightarrow{\mathbf{1}} \underline{0}{+}1{+}1 \xrightarrow{\mathbf{1}} \underline{1}{+}1 \xrightarrow{\mathbf{1}} \underline{2}
\end{aligned}
$$

Example ($\mathtt{waittt} \triangleq \mathtt{letrec}\, f^{\mathtt{Bool}\Rightarrow\mathtt{Bool}}\, x^{\mathtt{Bool}} = \mathtt{if}(x, \mathtt{tt}, (f)\, x)$)

$$(\mathtt{waittt})\left(\frac{1}{2}\mathtt{Coin}\right) \xrightarrow{1} ((\lambda f.\lambda x.\, \mathtt{if}(x, \mathtt{tt}, (f)\, x))\, \mathtt{waittt})\left(\frac{1}{2}\mathtt{Coin}\right)$$

$$\xrightarrow{1} (\lambda x.\mathtt{if}(x, \mathtt{tt}, (\mathtt{waittt})\, x))\left(\frac{1}{2}\mathtt{Coin}\right)$$

$$\xrightarrow{1} \mathtt{if}(\left(\frac{1}{2}\mathtt{Coin}\right), \mathtt{tt}, (\mathtt{waittt})\left(\frac{1}{2}\mathtt{Coin}\right))$$

$$\xrightarrow{\frac{1}{2}} \mathtt{if}(\mathtt{Coin}, \mathtt{tt}, (\mathtt{waittt})\left(\frac{1}{2}\mathtt{Coin}\right))$$

- $\xrightarrow{1} \mathtt{if}(\mathtt{tt}, \mathtt{tt}, (\mathtt{waittt})\left(\frac{1}{2}\mathtt{Coin}\right))$

  $\xrightarrow{1} \mathtt{tt}$

- $\xrightarrow{1} \mathtt{if}(\mathtt{ff}, \mathtt{tt}, (\mathtt{waittt})\left(\frac{1}{2}\mathtt{Coin}\right))$

  $\xrightarrow{1} (\mathtt{waittt})\left(\frac{1}{2}\mathtt{Coin}\right)$

Operational Semantics – Big Step

Definition

Let M be a closed term of ground type (i.e. `Bool`, `Int`),
let $V \in \{\mathtt{tt}, \mathtt{ff}, \underline{n}\}$, define

$$M \Downarrow^{\kappa} V \quad \text{iff} \quad \kappa = \sum_{M \xrightarrow{\kappa_1} \dots \xrightarrow{\kappa_n} V} \prod_{i=1}^{n} \kappa_i$$

- $\forall M \in \mathrm{PCF}$, the correspondence $M \Downarrow^1 V$ is a partial function:

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Theorem (Turing completeness)

A partial function $\phi : \mathbb{N} \rightharpoonup \mathbb{N}$ *is partial recursive iff there is a closed* PCF *term* $M : \mathtt{Int} \Rightarrow \mathtt{Int}$ *such that*

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Proof.

$\Rightarrow$ minimization is given by `letrec`,

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Operational Semantics – Small Step

Example ($\mathtt{or} \triangleq \lambda x^{\mathtt{Bool}}.\lambda y^{\mathtt{Bool}}.\mathtt{if}(x, \mathtt{tt}, y)$)

$$((\mathtt{or})\mathtt{tt})\,\mathtt{ff} \xrightarrow{1} (\lambda^{\mathtt{Bool}} y.\,\mathtt{if}(\mathtt{tt}, \mathtt{tt}, y))\,\mathtt{ff}$$

$$\xrightarrow{1} \mathtt{If}(\mathtt{tt}, \mathtt{tt}, \mathtt{ff})$$

$$\xrightarrow{1} \mathtt{tt}$$

Redex-to-contractum rules

$$\left(\lambda x^A.M\right) N \xrightarrow{1} M\{N/x\} \qquad \mathtt{fix}(M) \xrightarrow{1} (M)\,\mathtt{fix}(M)$$

$$\mathtt{iszero}(\underline{0}) \xrightarrow{1} \mathtt{tt} \qquad \mathtt{iszero}(\underline{n+1}) \xrightarrow{1} \mathtt{ff} \qquad \underline{n}{+}1 \xrightarrow{1} \underline{n+1} \qquad \underline{n+1}{-}1 \xrightarrow{1} \underline{n}$$

$$\mathtt{if}(\mathtt{tt}, M, N) \xrightarrow{1} M \qquad \mathtt{if}(\mathtt{ff}, M, N) \xrightarrow{1} N \qquad \mathtt{Coin} \xrightarrow{1} \mathtt{tt} \qquad \mathtt{Coin} \xrightarrow{1} \mathtt{ff} \qquad \kappa M \xrightarrow{\kappa} M$$

Contextual rules — supposing $M \xrightarrow{\kappa} N$

$$\text{if } M \text{ not a } \lambda,\ (M)\,P \xrightarrow{\kappa} (N)\,P \qquad \mathtt{iszero}(M) \xrightarrow{\kappa} \mathtt{iszero}(N)$$

$$M{+}1 \xrightarrow{\kappa} N{+}1 \qquad M{-}1 \xrightarrow{\kappa} N{-}1 \qquad \mathtt{if}(M, P, P') \xrightarrow{1} \mathtt{if}(N, P, P')$$

Operational Semantics – Small Step

Example ($\Omega \triangleq \mathtt{fix}(\lambda x^A.x)$)

$$\Omega \xrightarrow{1} \left(\lambda x^A.x\right) \mathtt{fix}(\lambda x^A.x)$$
$$\xrightarrow{1} \Omega$$

Example ($\frac{1}{2}\mathtt{Coin}$)

$$\frac{1}{2}\mathtt{Coin} \xrightarrow{\frac{1}{2}} \mathtt{Coin}$$
$$\bullet \xrightarrow{1} \mathtt{tt}$$
$$\bullet \xrightarrow{1} \mathtt{ff}$$

Operational Semantics – Small Step

Example ($\mathrm{double} \stackrel{\triangle}{=} \texttt{letrec}\, f^{\texttt{Int}\Rightarrow\texttt{Int}}\, x^{\texttt{Int}} = \texttt{if}(\mathrm{iszero}(x), \underline{0}, ((f)\,(x-1))+1+1))$

$$
\begin{aligned}
(\mathrm{double})\,\underline{1} &\xrightarrow{\mathbf{1}} ((\lambda f.\lambda x.\, \texttt{if}(\mathrm{iszero}(x), \underline{0}, ((f)\,(x-1))+1+1))\mathrm{double})\underline{1} \\
&\xrightarrow{\mathbf{1}} (\lambda x.\, \texttt{if}(\mathrm{iszero}(x), \underline{0}, ((\mathrm{double})\,(x-1))+1+1))\underline{1} \\
&\xrightarrow{\mathbf{1}} \texttt{if}(\mathrm{iszero}(\underline{1}), \underline{0}, ((\mathrm{double})\,(\underline{1}-1))+1+1) \\
&\xrightarrow{\mathbf{1}} \texttt{if}(\mathrm{ff}, \underline{0}, ((\mathrm{double})\,(\underline{1}-1))+1+1) \\
&\xrightarrow{\mathbf{1}} ((\mathrm{double})\,(\underline{1}-1))+1+1 \\
&\xrightarrow{\mathbf{1}} ((\lambda f.\lambda x.\, \texttt{if}(\mathrm{iszero}(x), \underline{0}, ((f)\,(x-1))+1+1))\mathrm{double}))\,(\underline{1}-1)+1+1 \\
&\xrightarrow{\mathbf{1}} ((\lambda x.\, \texttt{if}(\mathrm{iszero}(x), \underline{0}, ((\mathrm{double})\,(x-1))+1+1)))\,(\underline{1}-1))+1+1 \\
&\xrightarrow{\mathbf{1}} (\,\texttt{if}(\mathrm{iszero}((\underline{1}-1)), \underline{0}, ((\mathrm{double})\,((\underline{1}-1)-1))+1+1)))+1+1 \\
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&\xrightarrow{\mathbf{1}} (\texttt{if}(\mathrm{tt}, \underline{0}, ((\mathrm{double})\,((\underline{1}-1)-1))+1+1)))+1+1 \xrightarrow{\mathbf{1}} \underline{0}+1+1 \xrightarrow{\mathbf{1}} \underline{1}+1 \xrightarrow{\mathbf{1}} \underline{2}
\end{aligned}
$$

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\bullet &\xrightarrow{1} \mathtt{if}(\mathtt{tt}, \mathtt{tt}, (\mathtt{waittt})\left(\frac{1}{2}\mathtt{Coin}\right))\\
&\xrightarrow{1} \mathtt{tt}\\
\bullet &\xrightarrow{1} \mathtt{if}(\mathtt{ff}, \mathtt{tt}, (\mathtt{waittt})\left(\frac{1}{2}\mathtt{Coin}\right))\\
&\xrightarrow{1} (\mathtt{waittt})\left(\frac{1}{2}\mathtt{Coin}\right)
\end{aligned}
$$

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Definition

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- $\forall M \in$ PCF, the correspondence $M \Downarrow^1 V$ is a partial function:

  terms $\mapsto$ values

Theorem (Turing completeness)

A partial function $\phi : \mathbb{N} \rightharpoonup \mathbb{N}$ *is partial recursive iff there is a closed* PCF *term* M : `Int` $\Rightarrow$ `Int` *such that*

$$\phi(m) = n \;\; iff \;\; (M)\,\underline{m} \Downarrow^1 \underline{n}$$

Proof.

$\Rightarrow$ minimization is given by `letrec`,

$\Leftarrow$ the evaluation $M \Downarrow^1 V$ is partial recursive, via Gödel-numbering of PCF. □

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$\Rightarrow$ minimization is given by `letrec`,

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Denotation of Types

Ground Types

$$[\![\texttt{Bool}]\!] \triangleq \{\texttt{tt}, \texttt{ff}\} \qquad \text{i.e. } \mathfrak{R}^{\texttt{Bool}} = \{(\kappa, \rho) \mid \kappa, \rho \in \mathfrak{R}\}$$

$$[\![\texttt{Int}]\!] \triangleq \mathbb{N} \qquad \text{i.e. } \mathfrak{R}^{\texttt{Int}} = \{(\kappa_n)_{n \in \mathbb{N}} \mid \kappa_n \in \mathfrak{R}\}$$

Higher order Types

$A \Rightarrow B = !A \multimap B$

$$[\![A \Rightarrow B]\!] = [\![!A \multimap B]\!] \triangleq \mathcal{M}_f([\![A]\!]) \times [\![B]\!]$$

Denotation of Types

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$(Px)_{\texttt{tt}} =$ **?**

Denotation of Types

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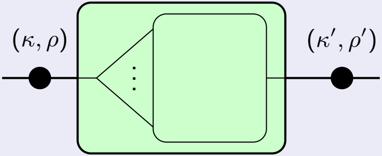

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Denotation of Types

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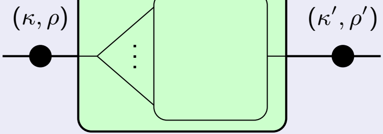

$$(Px)_{\texttt{tt}} = P_{[\,],\texttt{tt}} + P_{[\bullet],\texttt{tt}}x + P_{[\bullet,\bullet],\texttt{tt}}x + P_{[\bullet,\bullet,\bullet],\texttt{tt}}x \; \vdots$$

Denotation of Types

Ground Types

$[\![\texttt{Bool}]\!] \triangleq \{\texttt{tt}, \texttt{ff}\}$ i.e. $\mathfrak{R}^{\texttt{Bool}} = \{(\kappa, \rho) \mid \kappa, \rho \in \mathfrak{R}\}$

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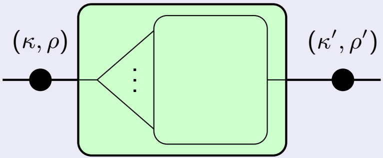

$$\begin{aligned}(Px)_{\texttt{tt}} = {} & P_{[\,],\texttt{tt}} \\ & + P_{[\texttt{tt}],\texttt{tt}}\, x_{\texttt{tt}} + P_{[\texttt{ff}],\texttt{tt}}\, x_{\texttt{ff}} \\ & + P_{[\bullet,\bullet],\texttt{tt}}\, x \\ & + P_{[\bullet,\bullet,\bullet],\texttt{tt}}\, x \\ & \vdots\end{aligned}$$

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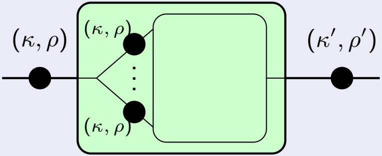

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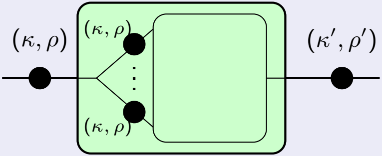

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Denotation of Types

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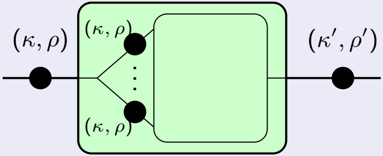

$$\begin{aligned}(Px)_{\texttt{tt}} = {} & P_{[\,],\texttt{tt}} \\ & + P_{[\texttt{tt}],\texttt{tt}} x_{\texttt{tt}} + P_{[\texttt{ff}],\texttt{tt}} x_{\texttt{ff}} \\ & + P_{(\texttt{tt},\texttt{tt}),\texttt{tt}} x_{\texttt{tt}}^2 + (P_{(\texttt{tt},\texttt{ff}),\texttt{tt}} + P_{(\texttt{ff},\texttt{tt}),\texttt{tt}}) x_{\texttt{tt}} x_{\texttt{ff}} + P_{(\texttt{ff},\texttt{ff}),\texttt{tt}} x_{\texttt{ff}}^2 \\ & + P_{[\bullet,\bullet,\bullet],\texttt{tt}} x \\ & \vdots\end{aligned}$$

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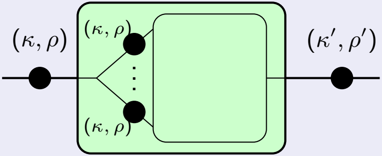

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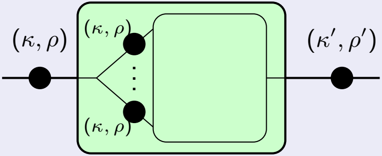
$A \Rightarrow B = !A \multimap B$

$$[\![A \Rightarrow B]\!] = [\![!A \multimap B]\!] \triangleq \mathcal{M}_f([\![A]\!]) \times [\![B]\!]$$



$$\begin{aligned}(Px)_{\texttt{tt}} = {} & P_{[\,],\texttt{tt}} \\ & + P_{[\texttt{tt}],\texttt{tt}} x_{\texttt{tt}} + P_{[\texttt{ff}],\texttt{tt}} x_{\texttt{ff}} \\ & + P_{[\texttt{tt},\texttt{tt}],\texttt{tt}} x_{\texttt{tt}}^2 + P_{[\texttt{tt},\texttt{ff}],\texttt{tt}} x_{\texttt{tt}} x_{\texttt{ff}} + P_{[\texttt{ff},\texttt{ff}],\texttt{tt}} x_{\texttt{ff}}^2 \\ & + P_{[\texttt{tt},\texttt{tt},\texttt{tt}],\texttt{tt}} x_{\texttt{tt}}^3 + P_{[\texttt{tt},\texttt{tt},\texttt{ff}],\texttt{tt}} x_{\texttt{tt}}^2 x_{\texttt{ff}} + P_{[\texttt{tt},\texttt{ff},\texttt{ff}],\texttt{tt}} x_{\texttt{tt}} x_{\texttt{ff}}^2 + P_{[\texttt{ff},\texttt{ff},\texttt{ff}],\texttt{tt}} x_{\texttt{ff}}^3 \\ & \vdots \\ = {} & \sum_{m \in \mathcal{M}_f(\{\texttt{tt},\texttt{ff}\})} P_{m,\texttt{tt}} x_{\texttt{tt}}^{m(\texttt{tt})} x_{\texttt{ff}}^{m(\texttt{ff})}\end{aligned}$$

$\Leftarrow$ power series in the unknowns $x_{\texttt{tt}}$ and $x_{\texttt{ff}}$.

Denotation of Terms

Let $\Gamma = x_1 : A_1, \ldots, x_n : A_n$, a typing judgement $\Gamma \vdash M : B$ is interpreted by $[\![M]\!]^{\Gamma}$,

a matrix in $\mathfrak{R}^{\prod_i \mathcal{M}_f(A_i) \times B}$ i.e. a power series $\prod_i \mathfrak{R}^{[\![A_i]\!]} \mapsto \mathfrak{R}^{B}$

$$[\![\mathtt{tt}]\!]^{\Gamma}_{r,b} \triangleq \begin{cases} \mathbf{1} & \text{if } r = [\,],\, b = \mathtt{tt}, \\ \mathbf{0} & \text{otherwise.} \end{cases} \qquad u \mapsto (\mathbf{1}, \mathbf{0})$$

$$[\![\mathtt{ff}]\!]^{\Gamma}_{r,b} \triangleq \begin{cases} \mathbf{1} & \text{if } r = [\,],\, b = \mathtt{ff}, \\ \mathbf{0} & \text{otherwise.} \end{cases} \qquad u \mapsto (\mathbf{0}, \mathbf{1})$$

$$[\![\mathtt{if}(N, P, Q)]\!]^{\Gamma}_{r,b} \triangleq \sum_{\substack{(r_1, r_2) \\ r_1 + r_2 = r}} \begin{pmatrix} [\![N]\!]^{\Gamma}_{r_1,\mathtt{tt}} \cdot [\![P]\!]^{\Gamma}_{r_2,b} \\ + [\![N]\!]^{\Gamma}_{r_1,\mathtt{ff}} \cdot [\![Q]\!]^{\Gamma}_{r_2,b} \end{pmatrix} \qquad u \mapsto [\![N]\!](u)_{\mathtt{tt}} \cdot [\![P]\!](u) + [\![N]\!](u)_{\mathtt{ff}} \cdot [\![Q]\!](u)$$

Denotation of Terms

Let $\Gamma = x_1 : A_1, \ldots, x_n : A_n$, a typing judgement $\Gamma \vdash M : B$ is interpreted by $[\![M]\!]^\Gamma$,

a matrix in $\mathfrak{R}^{\prod_i \mathcal{M}_f(A_i) \times B}$ i.e. a power series $\prod_i \mathfrak{R}^{[\![A_i]\!]} \mapsto \mathfrak{R}^B$

$$[\![\mathtt{tt}]\!]^\Gamma_{r,b} \triangleq \begin{cases} \mathbf{1} & \text{if } r = [\,],\ b = \mathtt{tt}, \\ \mathbf{0} & \text{otherwise.} \end{cases} \qquad u \mapsto (\mathbf{1}, \mathbf{0})$$

$$[\![\mathtt{ff}]\!]^\Gamma_{r,b} \triangleq \begin{cases} \mathbf{1} & \text{if } r = [\,],\ b = \mathtt{ff}, \\ \mathbf{0} & \text{otherwise.} \end{cases} \qquad u \mapsto (\mathbf{0}, \mathbf{1})$$

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$$[\![\mathtt{Coin}]\!]^{\Gamma}_{r,b} \stackrel{\triangle}{=} \begin{cases} \mathbf{1} & \text{if } r = [\,], b = \mathtt{tt}, \mathtt{ff}, \\ \mathbf{0} & \text{otherwise.} \end{cases} \qquad u \mapsto (\mathbf{1}, \mathbf{1})$$

$$[\![\kappa M]\!]^{\Gamma}_{r,b} \stackrel{\triangle}{=} \kappa\, [\![M]\!]^{\Gamma}_{r,b} \qquad u \mapsto \kappa\, [\![M]\!]\,(u)$$

$$[\![\underline{n}]\!]^{\Gamma}_{r,b} \stackrel{\triangle}{=} \begin{cases} \mathbf{1} & \text{if } r = [\,],\ b = n, \\ \mathbf{0} & \text{otherwise.} \end{cases} \qquad u \mapsto e_n$$

$$[\![P{-}1]\!]^{\Gamma}_{r,b} \stackrel{\triangle}{=} [\![P]\!]^{\Gamma}_{r,b+1} \qquad u \mapsto \big([\![P]\!]\,(u)_1, [\![P]\!]\,(u)_2, [\![P]\!]\,(u)_3, \dots\big)$$

$$[\![P{+}1]\!]^{\Gamma}_{r,b} \stackrel{\triangle}{=} [\![P]\!]^{\Gamma}_{r,b-1} \qquad u \mapsto \big(\mathbf{0}, [\![P]\!]\,(u)_0, [\![P]\!]\,(u)_1, \dots\big)$$

$$[\![\mathtt{iszero}(P)]\!]^{\Gamma}_{r,b} \stackrel{\triangle}{=} \Big([\![P]\!]^{\Gamma}_{r,0}, \sum_{n>0} [\![P]\!]^{\Gamma}_{r,n}\Big) \qquad u \mapsto \Big([\![P]\!]\,(u)_0, \sum_{n>0} [\![P]\!]^{\Gamma}\,(u)_n\Big)$$

Denotation of Terms

$$[\![x]\!]^{x:A,\Gamma}_{(m,r),a} \stackrel{\triangle}{=} \begin{cases} \mathbf{1} & \text{if } m = [a], \\ \mathbf{0} & \text{otherises.} \end{cases} \qquad v, u \mapsto v$$

$$\left[\!\!\left[\lambda x^A.M\right]\!\!\right]^{\Gamma}_{r,(m,b)} \stackrel{\triangle}{=} [\![M]\!]^{\Gamma,x:A}_{(r,m),b)} \qquad u \mapsto \mathrm{Mat}\left(v \mapsto [\![M]\!]\,(u,v)\right)$$

$$[\![(M)\,N]\!]^{\Gamma}_{r,b} \stackrel{\triangle}{=} \sum_{m \in \mathcal{M}_{\mathrm{f}}(A)} \sum_{\substack{(r_0,\dots,r_n) \\ \biguplus_i r_i = r}} [\![M]\!]^{\Gamma}_{r_0,(m,b)} \prod_{i=1}^{n} [\![N]\!]^{\Gamma}_{r_i,a_i} \qquad v \mapsto [\![M]\!]\,(v, [\![N]\!]\,(v))$$

where $m = [a_1, \dots, a_n]$

$$[\![\mathtt{fix}(M)]\!]^{\Gamma}_{r,b} \stackrel{\triangle}{=} \sup_{n=0}^{\infty} \left[\!\left[\mathtt{fix}^n(M)\right]\!\right]^{\Gamma,x:A}_{(r,m),b} \qquad u \mapsto \sup_{n=0}^{\infty} \underbrace{[\![M]\!]\,(u, (\dots([\![M]\!]}_{n \text{ times}}(u,\mathbf{0}))))$$

Denotation of Terms

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Denotation of Terms

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ete partial order on $\mathfrak{R}$:
ommutative semi-ring

Example ($\mathtt{or} \triangleq \lambda x^{\mathtt{Bool}}.\lambda y^{\mathtt{Bool}}.\mathtt{if}(x, \mathtt{tt}, y)$)

$$\llbracket \mathtt{or}\, xy \rrbracket_{(m,p),\mathtt{tt}} = \begin{cases} \mathbf{1} & \text{if } m = [\mathtt{tt}],\, p = [\,], \\ \mathbf{1} & \text{if } m = [\mathtt{ff}],\, p = [\mathtt{tt}], \\ \mathbf{0} & \text{otherwise.} \end{cases} \qquad (\mathtt{or}\, xy)_{\mathtt{tt}} = x_{\mathtt{tt}} + x_{\mathtt{ff}}\, y_{\mathtt{tt}}$$

$$\llbracket \mathtt{or}\, xy \rrbracket_{(m,p),\mathtt{ff}} = \begin{cases} \mathbf{1} & \text{if } m = [\mathtt{ff}],\, p = [\mathtt{ff}], \\ \mathbf{0} & \text{otherwise.} \end{cases} \qquad (\mathtt{or}\, xy)_{\mathtt{ff}} = x_{\mathtt{ff}}\, y_{\mathtt{ff}}$$

Example ($\Omega \triangleq \mathtt{fix}(\lambda x^A.x)$)

$$\llbracket \Omega \rrbracket = \mathbf{0}$$

Example ($\frac{1}{2}\mathtt{Coin}$)

$$\left\llbracket \frac{1}{2}\mathtt{Coin} \right\rrbracket = (\frac{1}{2}, \frac{1}{2})$$

Example ($\mathtt{or} \triangleq \lambda x^{\mathtt{Bool}}.\lambda y^{\mathtt{Bool}}.\mathtt{if}(x, \mathtt{tt}, y)$)

$$[\![\mathtt{or}\, xy]\!]_{(m,p),\mathtt{tt}} = \begin{cases} \mathbf{1} & \text{if } m = [\mathtt{tt}], p = [\,], \\ \mathbf{1} & \text{if } m = [\mathtt{ff}], p = [\mathtt{tt}], \\ \mathbf{0} & \text{otherwise.} \end{cases} \qquad (\mathtt{or}\, xy)_{\mathtt{tt}} = x_{\mathtt{tt}} + x_{\mathtt{ff}}\, y_{\mathtt{tt}}$$

$$[\![\mathtt{or}\, xy]\!]_{(m,p),\mathtt{ff}} = \begin{cases} \mathbf{1} & \text{if } m = [\mathtt{ff}], p = [\mathtt{ff}], \\ \mathbf{0} & \text{otherwise.} \end{cases} \qquad (\mathtt{or}\, xy)_{\mathtt{ff}} = x_{\mathtt{ff}}\, y_{\mathtt{ff}}$$

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$$[\![\Omega]\!] = \mathbf{0}$$

Example ($\frac{1}{2}\mathtt{Coin}$)

$$\left[\!\!\left[\frac{1}{2}\mathtt{Coin}\right]\!\!\right] = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Denotation of Terms

Example ($\mathtt{or} \triangleq \lambda x^{\mathtt{Bool}}.\lambda y^{\mathtt{Bool}}.\mathtt{if}(x, \mathtt{tt}, y)$)

$$[\![\mathtt{or}\,xy]\!]_{(m,p),\mathtt{tt}} = \begin{cases} \mathbf{1} & \text{if } m = [\mathtt{tt}],\ p = [\,], \\ \mathbf{1} & \text{if } m = [\mathtt{ff}],\ p = [\mathtt{tt}], \\ \mathbf{0} & \text{otherwise.} \end{cases} \qquad (\mathtt{or}\,xy)_{\mathtt{tt}} = x_{\mathtt{tt}} + x_{\mathtt{ff}}\,y_{\mathtt{tt}}$$

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$$[\![\Omega]\!] = \mathbf{0}$$

Example ($\frac{1}{2}\mathtt{Coin}$)

$$\left[\!\!\left[\frac{1}{2}\mathtt{Coin}\right]\!\!\right] = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Example ($\mathrm{double} \stackrel{\triangle}{=} \mathtt{letrec}\ f^{\mathtt{Int}\Rightarrow\mathtt{Int}}\ x^{\mathtt{Int}} = \mathtt{if}(\mathtt{iszero}(x), \underline{0}, ((f)\,(x{-}1)){+}1{+}1))$

$$[\![\mathrm{double}\,x]\!]_{m,n} = \begin{cases} \mathbf{1} & \text{if } m = [0],\ n = 0, \\ \binom{s}{h_1,\dots,h_r} & \text{if } m = [k_1^{h_1}, \dots, k_r^{h_r}, s], \text{each } k_i \geq \sum_{j<i} h_j, \\ & s = \sum_i h_i > 0,\ i \neq j \text{ implies } k_i \neq k_j, \\ & \text{and } n = 2s, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$(\mathrm{double}\,x)_{\underline{0}} = x_0$$

$$(\mathrm{double}\,x)_{\underline{2}} = \sum_{k=1}^{\infty} x_1 x_k$$

$$(\mathrm{double}\,x)_{\underline{4}} = x_1 \Big(\sum_{k=1}^{\infty} x_k^2 + \sum_{k_1=1}^{\infty} \sum_{\substack{k_2=1 \\ k_2 \neq k_1}}^{\infty} \mathbf{2} x_{k_1} x_{k_2}\Big) \qquad \text{where } \mathbf{2} = \mathbf{1} + \mathbf{1}$$

Example ($\mathtt{waittt} \triangleq \mathtt{letrec}\ f^{\mathtt{Bool}\Rightarrow\mathtt{Bool}}\ x^{\mathtt{Bool}} = \mathtt{if}(x, \mathtt{tt}, (f)\,x)$)

$$[\![\mathtt{waittt}\, x]\!]_{m,\mathtt{tt}} = \begin{cases} \mathbf{1} & \text{if } m = [\mathtt{tt}, \mathtt{ff}, \ldots, \mathtt{ff}], \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

$$(\mathtt{waittt}\, x)_{\mathtt{tt}} = \sum_{k=0}^{\infty} x_{\mathtt{tt}}\, x_{\mathtt{ff}}^{k}$$

Notice that, taking $\mathfrak{R} = (\overline{\mathbb{R}^+}, \cdot, 1, \sum, \leq)$:

$$(\mathtt{waittt}(\tfrac{1}{2}\mathtt{Coin}))_{\mathtt{tt}} = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = 1 \qquad (\mathtt{waittt}(\mathtt{Coin}))_{\mathtt{tt}} = \sum_{k=0}^{\infty} 1 = \infty$$

Example ($\mathtt{waittt} \stackrel{\triangle}{=} \mathtt{letrec}\, f^{\mathtt{Bool}\Rightarrow\mathtt{Bool}}\, x^{\mathtt{Bool}} = \mathtt{if}(x, \mathtt{tt}, (f)\, x)$)

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Road Map

- Weighted relational semantics
  - the category $\mathfrak{R}^{\Pi}$
  - $\mathfrak{R}^{\Pi}$ is a model of Linear Logic
- A prototypical functional language
  - types and terms
  - operational semantics
  - denotational semantics $\mathfrak{R}^{\Pi}_{!}$
- What can we observe?
  - parametric adequacy
  - some instances

G. Plotkin
LCF considered as a programming language
Theoretical Computer Science, 1975.

J.-Y. Girard.
Normal Functors, power series and lambda-calculus
Annals of Pure and Applied Logic, 1988.

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V. Danos and T. Ehrhard.
Probabilistic coherence spaces as a model of higher-order probabilistic computation.
Information & Computation, 2011.

J. Laird, G. Manzonetto, G. McCusker, M. Pagani.
Weighted Relational Models of Typed Lambda-calculus.
Proceedings of LICS, 2013.

Parametrized Quantitative Results for PCF$^{\mathfrak{R}}$

Theorem (Adequacy)

For every closed $M \in \mathrm{PCF}^{\mathfrak{R}}$ of ground type (i.e. `Bool`, `Int`*),*

$$\forall \text{ value } V, [\![M]\!]_V = \kappa \quad \textit{iff} \quad M \Downarrow^{\kappa} V$$

What are we actually counting ?

This depends on the chosen semi-ring $\mathfrak{R}$ and fragment of PCF$^{\mathfrak{R}}$

$$x \mid \lambda x^A.P \mid (P)\,Q \mid \mathtt{fix}(P) \mid \underline{n} \mid P{-}1 \mid P{+}1 \mid \mathtt{iszero}(P) \mid \mathtt{tt} \mid \mathtt{ff} \mid \mathtt{if}(N, P, Q) \mid \mathtt{Coin}$$

$\mathfrak{R}$	$[\![M]\!]_V$

and more... see [Laird-Manzonetto-McCusker-Pagani, 2013]

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$\mathfrak{R}$	$[\![M]\!]_V$
$(\overline{\mathbb{N}}, \cdot, 1, \sum, \leq)$	number of paths from M to V

and more… see [Laird-Manzonetto-McCusker-Pagani, 2013]

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$\mathfrak{R}$	$[\![M]\!]_V$
$(\overline{\mathbb{N}}, \cdot, 1, \sum, \leq)$	number of paths from M to V
$(\overline{\mathbb{R}^+}, \cdot, 1, \sum, \leq)$	probability that M reduces to V [Danos-Ehrhard'11]

and more… see [Laird-Manzonetto-McCusker-Pagani, 2013]

Theorem (Adequacy)

For every closed $M \in \mathrm{PCF}^{\mathfrak{R}}$ *of ground type (i.e.* $\texttt{Bool}, \texttt{Int}$*),*

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What are we actually counting ?

This depends on the chosen semi-ring $\mathfrak{R}$ and fragment of PCF$^{\mathfrak{R}}$

$$x \mid \lambda x^A.1P \mid (P)\,Q \mid \mathtt{fix}(P) \mid \underline{n} \mid P{-}1 \mid P{+}1 \mid \mathtt{iszero}(P) \mid \mathtt{tt} \mid \mathtt{ff} \mid \mathtt{if}(N, P, Q) \mid \mathtt{Coin}$$

$\mathfrak{R}$	$[\![M]\!]_V$
$(\overline{\mathbb{N}}, \cdot, 1, \sum, \leq)$	number of paths from M to V
$(\overline{\mathbb{R}^+}, \cdot, 1, \sum, \leq)$	probability that M reduces to V [Danos-Ehrhard'11]
$(\overline{\mathbb{N}}, +, 0, \min, \geq)$	minimum number of β steps to V

and more. . . see [Laird-Manzonetto-McCusker-Pagani, 2013]

Example ($\mathrm{double} \overset{\triangle}{=} \mathtt{letrec}\, f^{\mathtt{Int}\Rightarrow\mathtt{Int}}\, x^{\mathtt{Int}} = \mathtt{if}(\mathrm{iszero}(x), \underline{0}, ((f)\,(x-1))+1+1)$)

$$
\begin{aligned}
(\mathrm{double})\,\underline{1} &\xrightarrow{1} ((\lambda f.\lambda x.\,\mathtt{if}(\mathrm{iszero}(x), \underline{0}, ((f)\,(x-1))+1+1))\mathrm{double})\underline{1}\\
&\xrightarrow{1} (\lambda x.\,\mathtt{if}(\mathrm{iszero}(x), \underline{0}, ((\mathrm{double})\,(x-1))+1+1))\underline{1}\\
&\xrightarrow{1} \mathtt{if}(\mathrm{iszero}(\underline{1}), \underline{0}, ((\mathrm{double})\,(\underline{1}-1))+1+1)\\
&\xrightarrow{1} \mathtt{if}(\mathtt{ff}, \underline{0}, ((\mathrm{double})\,(\underline{1}-1))+1+1)\\
&\xrightarrow{1} ((\mathrm{double})\,(\underline{1}-1))+1+1\\
&\xrightarrow{1} ((\lambda f.\lambda x.\,\mathtt{if}(\mathrm{iszero}(x), \underline{0}, ((f)\,(x-1))+1+1))\mathrm{double}))\,(\underline{1}-1)+1+1\\
&\xrightarrow{1} ((\lambda x.\,\mathtt{if}(\mathrm{iszero}(x), \underline{0}, ((\mathrm{double})\,(x-1))+1+1)))\,(\underline{1}-1))+1+1\\
&\xrightarrow{1} (\,\mathtt{if}(\mathrm{iszero}((\underline{1}-1)), \underline{0}, ((\mathrm{double})\,((\underline{1}-1)-1))+1+1)))+1+1\\
&\xrightarrow{1} (\mathtt{if}(\mathrm{iszero}(\underline{0}), \underline{0}, ((\mathrm{double})\,((\underline{1}-1)-1))+1+1)))+1+1\\
&\xrightarrow{1} (\mathtt{if}(\mathtt{tt}, \underline{0}, ((\mathrm{double})\,((\underline{1}-1)-1))+1+1)))+1+1 \xrightarrow{1} \underline{0}+1+1 \xrightarrow{1} \underline{1}+1 \xrightarrow{1} \underline{2}
\end{aligned}
$$

Example ($\mathrm{double} \stackrel{\triangle}{=} \mathtt{letrec}\, f^{\mathtt{Int}\Rightarrow\mathtt{Int}}\, x^{\mathtt{Int}} = \mathtt{if}(\mathtt{iszero}(x), \underline{0}, ((f)\,(x{-}1)){+}1{+}1)$)

$$
\begin{aligned}
(\mathrm{double})\,\underline{1} &\xrightarrow{0} ((\lambda f.\lambda x.\,\mathtt{if}(\mathtt{iszero}(x), \underline{0}, ((f)\,(x{-}1)){+}1{+}1))\mathrm{double})\underline{1} \\
&\xrightarrow{0} (\lambda x.\,\mathtt{if}(\mathtt{iszero}(x), \underline{0}, ((\mathrm{double})\,(x{-}1)){+}1{+}1))\underline{1} \\
&\xrightarrow{0} \mathtt{if}(\mathtt{iszero}(\underline{1}), \underline{0}, ((\mathrm{double})\,(\underline{1}{-}1)){+}1{+}1) \\
&\xrightarrow{0} \mathtt{if}(\mathtt{ff}, \underline{0}, ((\mathrm{double})\,(\underline{1}{-}1)){+}1{+}1) \\
&\xrightarrow{0} ((\mathrm{double})\,(\underline{1}{-}1)){+}1{+}1 \\
&\xrightarrow{0} ((\lambda f.\lambda x.\,\mathtt{if}(\mathtt{iszero}(x), \underline{0}, ((f)\,(x{-}1)){+}1{+}1))\mathrm{double}))\,(\underline{1}{-}1){+}1{+}1 \\
&\xrightarrow{0} ((\lambda x.\,\mathtt{if}(\mathtt{iszero}(x), \underline{0}, ((\mathrm{double})\,(x{-}1)){+}1{+}1)))\,(\underline{1}{-}1)){+}1{+}1 \\
&\xrightarrow{0} (\,\mathtt{if}(\mathtt{iszero}((\underline{1}{-}1)), \underline{0}, ((\mathrm{double})\,((\underline{1}{-}1){-}1)){+}1{+}1))){+}1{+}1 \\
&\xrightarrow{0} (\mathtt{if}(\mathtt{iszero}(\underline{0}), \underline{0}, ((\mathrm{double})\,((\underline{1}{-}1){-}1)){+}1{+}1))){+}1{+}1 \\
&\xrightarrow{0} (\mathtt{if}(\mathtt{tt}, \underline{0}, ((\mathrm{double})\,((\underline{1}{-}1){-}1)){+}1{+}1))){+}1{+}1 \xrightarrow{0} \underline{0}{+}1{+}1 \xrightarrow{0} \underline{1}{+}1 \xrightarrow{0} \underline{2}
\end{aligned}
$$

Example ($\mathrm{double} \triangleq \mathtt{letrec}\, f^{\mathtt{Int}\Rightarrow\mathtt{Int}}\, x^{\mathtt{Int}} = \mathtt{if}(\mathtt{iszero}(x), \underline{0}, ((f)\,(x{-}1)){+}1{+}1)$)

$$
\begin{aligned}
(\overline{\mathrm{double}})\,\underline{1} &\xrightarrow{0} ((\lambda f.{\color{red}1}\lambda x.{\color{red}1}\,\mathtt{if}(\mathtt{iszero}(x),\underline{0},((f)\,(x{-}1)){+}1{+}1))\overline{\mathrm{double}})\underline{1} \\
&\xrightarrow{0} ({\color{red}1}\lambda x.{\color{red}1}\,\mathtt{if}(\mathtt{iszero}(x),\underline{0},((\overline{\mathrm{double}})\,(x{-}1)){+}1{+}1))\underline{1} \\
&\xrightarrow{0} {\color{red}1}\,\mathtt{if}(\mathtt{iszero}(\underline{1}),\underline{0},((\overline{\mathrm{double}})\,(\underline{1}{-}1)){+}1{+}1) \\
&\xrightarrow{0} \mathtt{if}(\mathtt{ff},\underline{0},((\overline{\mathrm{double}})\,(\underline{1}{-}1)){+}1{+}1) \\
&\xrightarrow{0} ((\overline{\mathrm{double}})\,(\underline{1}{-}1)){+}1{+}1 \\
&\xrightarrow{0} ((\lambda f.{\color{red}1}\lambda x.{\color{red}1}\,\mathtt{if}(\mathtt{iszero}(x),\underline{0},((f)\,(x{-}1)){+}1{+}1))\overline{\mathrm{double}}))\,(\underline{1}{-}1){+}1{+}1 \\
&\xrightarrow{0} (({\color{red}1}\lambda x.{\color{red}1}\,\mathtt{if}(\mathtt{iszero}(x),\underline{0},((\overline{\mathrm{double}})\,(x{-}1)){+}1{+}1)))\,(\underline{1}{-}1)){+}1{+}1 \\
&\xrightarrow{0} ({\color{red}1}\,\mathtt{if}(\mathtt{iszero}((\underline{1}{-}1)),\underline{0},((\overline{\mathrm{double}})\,((\underline{1}{-}1){-}1)){+}1{+}1))){+}1{+}1 \\
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&\xrightarrow{0} (\mathtt{if}(\mathtt{tt},\underline{0},((\overline{\mathrm{double}})\,((\underline{1}{-}1){-}1)){+}1{+}1))){+}1{+}1 \xrightarrow{0} \underline{0}{+}1{+}1 \xrightarrow{0} \underline{1}{+}1 \xrightarrow{0} \underline{2}
\end{aligned}
$$

Example $(\text{double} \stackrel{\triangle}{=} \texttt{letrec}\, f^{\texttt{Int}\Rightarrow\texttt{Int}}\, x^{\texttt{Int}} = \texttt{if}(\texttt{iszero}(x), \underline{0}, ((f)\,(x-1))+1+1))$

$$(\overline{\text{double}})\ \underline{1} \xrightarrow{0} ((\lambda f.1\lambda x.1\ \texttt{if}(\texttt{iszero}(x), \underline{0}, ((f)\,(x-1))+1+1))\overline{\text{double}})\underline{1}$$

$$\xrightarrow{0} (1\lambda x.1\ \texttt{if}(\texttt{iszero}(x), \underline{0}, ((\overline{\text{double}})\ (x-1))+1+1))\underline{1}$$

$$\xrightarrow{1} (\lambda x.1\ \texttt{if}(\texttt{iszero}(x), \underline{0}, ((\overline{\text{double}})\ (x-1))+1+1))\underline{1}$$

$$\xrightarrow{0} 1\ \texttt{if}(\texttt{iszero}(\underline{1}), \underline{0}, ((\overline{\text{double}})\ (\underline{1}-1))+1+1)$$

$$\xrightarrow{1} \texttt{if}(\texttt{iszero}(\underline{1}), \underline{0}, ((\overline{\text{double}})\ (\underline{1}-1))+1+1)$$

$$\xrightarrow{0} \texttt{if}(\texttt{ff}, \underline{0}, ((\overline{\text{double}})\ (\underline{1}-1))+1+1)$$

$$\xrightarrow{0} ((\overline{\text{double}})\ (\underline{1}-1))+1+1$$

$$\xrightarrow{0} ((\lambda f.1\lambda x.1\ \texttt{if}(\texttt{iszero}(x), \underline{0}, ((f)\,(x-1))+1+1))\overline{\text{double}}))\ (\underline{1}-1)+1+1$$

$$\xrightarrow{0} ((1\lambda x.1\ \texttt{if}(\texttt{iszero}(x), \underline{0}, ((\overline{\text{double}})\ (x-1))+1+1)))\ (\underline{1}-1))+1+1$$

$$\xrightarrow{1} ((\lambda x.1\ \texttt{if}(\texttt{iszero}(x), \underline{0}, ((\overline{\text{double}})\ (x-1))+1+1)))\ (\underline{1}-1))+1+1$$

$$\xrightarrow{0} (1\ \texttt{if}(\texttt{iszero}((\underline{1}-1)), \underline{0}, ((\overline{\text{double}})\ ((\underline{1}-1)-1))+1+1)))+1+1$$

$$\xrightarrow{1} (\texttt{if}(\texttt{iszero}((\underline{1}-1)), \underline{0}, ((\overline{\text{double}})\ ((\underline{1}-1)-1))+1+1)))+1+1$$

$$\xrightarrow{0} (\texttt{if}(\texttt{iszero}(\underline{0}), \underline{0}, ((\overline{\text{double}})\ ((\underline{1}-1)-1))+1+1)))+1+1$$

$$\xrightarrow{0} (\texttt{if}(\texttt{tt}, \underline{0}, ((\overline{\text{double}})\ ((\underline{1}-1)-1))+1+1)))+1+1 \xrightarrow{0} \underline{0}+1+1 \xrightarrow{0} \underline{1}+1 \xrightarrow{0} \underline{2}$$

Theorem (Adequacy)

For every closed $M \in \mathrm{PCF}^{\mathfrak{R}}$ of ground type (i.e. $\texttt{Bool}, \texttt{Int}$*),*

$$\forall \textit{ value } V, [\![M]\!]_V = \kappa \quad \textit{iff} \quad M \Downarrow^{\kappa} V$$

What are we actually counting ?

This depends on the chosen semi-ring $\mathfrak{R}$ and fragment of PCF$^{\mathfrak{R}}$

$$x \mid \lambda x^A.1P \mid (P)\,Q \mid \mathtt{fix}(P) \mid \underline{n} \mid P{-}1 \mid P{+}1 \mid \mathtt{iszero}(P) \mid \mathtt{tt} \mid \mathtt{ff} \mid \mathtt{if}(N, P, Q) \mid \mathtt{Coin}$$

$\mathfrak{R}$	$[\![M]\!]_V$
$(\overline{\mathbb{N}}, \cdot, 1, \sum, \leq)$	number of paths from M to V
$(\overline{\mathbb{R}^+}, \cdot, 1, \sum, \leq)$	probability that M reduces to V [Danos-Ehrhard'11]
$(\overline{\mathbb{N}}, +, 0, \min, \geq)$	minimum number of β steps to V

and more… see [Laird-Manzonetto-McCusker-Pagani, 2013]

Parametrized Quantitative Results for $\mathrm{PCF}^{\mathfrak{R}}$

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Parametrized Quantitative Results for $\mathrm{PCF}^{\mathfrak{R}}$

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Tutorial on Quantitative Semantics (II)

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Logoi Summer School, Torino 2013

Road Map

1. $\mathfrak{R}$-**Weighted** relational semantics for **Probabilisitic** programs
2. **Semantics** via **Intersection Type Systems**
3. **Work on** Examples

Modeling Probabilistic Data and Programs:

Type: set of positive vectors

Program: function seen as a positive matrix

Interaction: composition seen as multiplication

Modeling Probabilistic **Data**

Example: `nat`

$\frac{1}{2}$`Coin:Int` returns the toss of a fair coin.

`Random n:Int` returns uniformly any $\{0, \ldots, n-1\}$.

Non Determinism, a first approximation: $|\texttt{nat}| = \mathbb{N}$.

$$\left|\tfrac{1}{2}\texttt{Coin}\right| = \{0, 1\} \quad \text{and} \quad |\texttt{Random n}| = \{0, \ldots, n-1\}$$

Enriching with positive coefficients: $[\![\texttt{Int}]\!] \subseteq (\overline{\mathbb{R}^+})^{\mathbb{N}}$.

$$[\![\tfrac{1}{2}\texttt{Coin}]\!] = \Big(\underset{0}{\tfrac{1}{2}}, \underset{1}{\tfrac{1}{2}}, \underset{2}{0}, \underset{\ldots}{\ldots}\Big) \quad \text{and} \quad [\![\texttt{Random n}]\!] = \Big(\underset{0}{\tfrac{1}{n}}, \underset{\ldots}{\ldots}, \underset{n-1}{\tfrac{1}{n}}, 0, \underset{\ldots}{\ldots}\Big)$$

Subprobability Distributions over $\mathbb{N}$:

$$[\![\texttt{Int}]\!] = \left\{ (\lambda_n)_{n \in \mathbb{N}} \mid \forall n, \lambda_n \in \overline{\mathbb{R}^+} \text{ and } \sum_n \lambda_n \leq 1 \right\}$$

Modeling Probabilistic **Programs**

Example: $\texttt{Random} : \texttt{Int} \Rightarrow \texttt{Int}$

Input: an integer n

Output: any integer $\{0, \ldots, n-1\}$ uniformly chosen.

Non Determinism: $|\texttt{Random}| \subseteq |\texttt{Int}| \times |\texttt{Int}|$ is a relation.

$$|\texttt{Random}| = \{(n, k) \mid n \in \mathbb{N},\ 0 \leq k \leq n-1\}$$

Enriching with positive coefficients: $[\![\texttt{Random}]\!] \in (\overline{\mathbb{R}^+})^{(\mathbb{N}\times\mathbb{N})}$.

$$\begin{array}{c} \begin{array}{cccccc} \overset{0}{\downarrow} & \overset{1}{\downarrow} & \overset{2}{\downarrow} & \cdots & \overset{n}{\downarrow} & \cdots \end{array} \\ \begin{pmatrix} 0 & 1 & \frac{1}{2} & \cdots & \frac{1}{n} & \cdots \\ \vdots & 0 & \frac{1}{2} & \cdots & \frac{1}{n} & \cdots \\ & \vdots & 0 & \ddots & \vdots & \\ & & \vdots & 0 & \frac{1}{n} & \\ & & & \vdots & \ddots & \ddots \end{pmatrix} \begin{array}{l} \rightarrow 0 \\ \rightarrow 1 \\ \vdots \\ \rightarrow n-1 \\ \vdots \end{array} \end{array}$$

Modeling Probabilistic **Programs**

`Once : Int ⇒ Int`

Input: an integer `x`

Output:

```
if x=0
   then ½Coin
   else 42
```

$$\begin{array}{c} \begin{array}{cccc} 0 & 1 & \cdots & \cdots \\ \downarrow & \downarrow & & \end{array} \\ \begin{pmatrix} \frac{1}{2} & 0 & \cdots & \\ \frac{1}{2} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \cdots & 0 & \cdots & \ddots \\ 0 & 1 & 1 & \cdots \\ \cdots & 0 & \cdots & \ddots \end{pmatrix} \begin{array}{l} \to 0 \\ \to 1 \\ \vdots \\ \\ \to 42 \\ \vdots \end{array} \end{array}$$

`Twice : Int ⇒ Int`

Input: an integer `x`

Output:

```
if x=0
   then if x=0
           then ½Coin
           else 42
   else if x=0
           then 42
           else 0
```

$([0,0],\ 0)$	$\mapsto$	$\frac{1}{2}$	
$([0,0],\ 1)$	$\mapsto$	$\frac{1}{2}$	
$([0,a],42)$	$\mapsto$	2	if $0<a$
$([a,b],\ 0)$	$\mapsto$	2	if $0<a,b$
Otherwise		0	

Modeling Probabilistic **Interaction**

Probabilistic Data :
If $\mathtt{x} : \mathtt{Int}$, then $[\![\mathtt{x}]\!] = (x_a)_{a \in \mathbb{N}}$

where x_a is the probability that $\mathtt{x}$ is a.

Probabilistic Program : $\mathtt{P} : \mathtt{Int} \Rightarrow \mathtt{Int}$

where $[\![\mathtt{P}\ \mathtt{x}]\!]_b$ is the probability that $\mathtt{P}\ \mathtt{x}$ computes b.

$$[\![\mathtt{Once}]\!] \in (\overline{\mathbb{R}^+})^{\mathbb{N} \times \mathbb{N}}$$

$$[\![\mathtt{Once}\ \boldsymbol{x}]\!]_b = [\![\mathtt{Once}]\!] \cdot [\![\mathtt{x}]\!] = \sum_{a \in \mathbb{N}} [\![\mathtt{Once}]\!]_{(a,b)}\ [\![\mathtt{x}]\!]_a$$

$$[\![\mathtt{Twice}]\!] \in (\overline{\mathbb{R}^+})^{\mathcal{M}_f(\mathbb{N}) \times \mathbb{N}}$$

$$\begin{aligned}
[\![\mathtt{Twice}\ \boldsymbol{x}]\!]_b &= [\![\mathtt{Twice}]\!] \cdot [\![\mathtt{x}]\!]^! \\
&= \sum_{m \in \mathcal{M}_f(\mathbb{N})} [\![\mathtt{Twice}]\!]_{(m,b)} \prod_{a \in \mathrm{Supp}(m)} [\![\mathtt{x}]\!]_a^{m(a)} \\
&= \sum_{m \in \mathcal{M}_f(\mathbb{N})} [\![\mathtt{Twice}]\!]_{(m,b)} \prod_{(a,i) \in m} [\![\mathtt{x}]\!]_a \\
&= \sum_{a,a' \in \mathbb{N}} [\![\mathtt{Twice}]\!]_{([a,a'],b)}\ [\![\mathtt{x}]\!]_a\ [\![\mathtt{x}]\!]_{a'}
\end{aligned}$$

Weighted Relational Category over $\overline{\mathbb{R}^+}$

Objects: a countable set $|\mathcal{X}|$

Maps: $f : |\mathcal{X}| \to |\mathcal{Y}|$ defined as a **matrix** $\mathrm{Mat}(f) \in (\overline{\mathbb{R}^+})^{\mathcal{M}_f(|\mathcal{X}|) \times |\mathcal{Y}|}$

Composition Formula:

$$f(x) = \sum_{m \in \mathcal{M}_f(|\mathcal{X}|)} \mathrm{Mat}(f)_m \cdot x^m$$

with $x^m = \prod_{a \in \mathrm{Supp}(x)} x_a^{m(a)}$

f can be seen as an **entire function** $f : (\overline{\mathbb{R}^+})^{|\mathcal{X}|} \to (\overline{\mathbb{R}^+})^{|\mathcal{Y}|}$

Adequacy Theorem:

Let M : `Int` be a closed program. Then for all $a \in \mathbb{N}$,

$$M \Downarrow^{\kappa} a \iff \kappa \doteq \mathbf{Proba}(M \xrightarrow{*} \underline{a}) = [\![M]\!]_a .$$

Intersection Type system: De Carvalho's **System R** with coefficients

Syntactical types: $x_1 : C_1, \ldots, x_n : C_n \vdash M : A$

Semantical types : $\left(\mathcal{M}_{\mathbf{f}}\left(|\mathcal{C}_1|\right) \times \cdots \times \mathcal{M}_{\mathbf{f}}\left(|\mathcal{C}_n|\right)\right) \times |\mathcal{A}| \xrightarrow{[\![M]\!]} \overline{\mathbb{R}^+}$

if $[\![M]\!]_{(m_1,\ldots,m_n,a)} \neq 0$, then $x_1^{C_1} : m_1, \ldots, x_n^{C_n} : m_n \overset{\pi}{\vdash} M : a$

$$[\![M]\!]_{(m_1,\ldots,m_n,a)} = \sum_{\pi} \omega(\pi)$$

Remark: $(m, a) = ([c_1, \ldots, c_n], a) = c_1 \wedge \cdots \wedge c_n \to a$ is an intersection type.

Semantics and Intersection Type system

$$\frac{}{x^A : [a] \vdash_1 x : a}\ \text{var} \qquad \frac{}{\vdash_1 \underline{n} : n}\ \text{nat} \qquad \frac{k < n}{\vdash_{\frac{1}{n}} \texttt{Rand} : ([n], k)}\ \text{rand} \qquad \frac{\Gamma^\bullet, x^A : m \vdash_\alpha M : a}{\Gamma^\bullet \vdash_\alpha \lambda x^A.M : (m, a)}\ \text{abs}$$

$$\frac{\Gamma^{\bullet\prime} \vdash_\alpha M : (m, b) \qquad \forall (a,i) \in m,\quad \Gamma^\bullet_{(a,i)} \vdash_{\beta_{(a,i)}} N : a}{\Gamma^\bullet \vdash_{\alpha \prod_{(a,i)\in m} \beta_{(a,i)}} (M)\, N : b}\ \text{app}_{(m,(\Gamma^\bullet_{(a,i)})_{(a,i)\in m})} \quad \text{s.t.} \begin{cases} m \in \mathcal{M}_{\mathrm{f}}(|\mathcal{A}|) \\ \Gamma^{\bullet\prime} \uplus \biguplus_{(a,i)\in m} \Gamma^\bullet_{(a,i)} = \Gamma^\bullet \end{cases}$$

$$\frac{\Gamma^{\bullet\prime} \vdash_\alpha M : (m, b) \qquad \forall (a,i) \in m,\quad \Gamma^\bullet_{(a,i)} \vdash_{\beta_{(a,i)}} \mathtt{fix}(M) : a}{\Gamma^\bullet \vdash_{\alpha \prod_{(a,i)\in m} \beta_{(a,i)}} \mathtt{fix}(M) : b}\ \text{fix}_{(m,(\Gamma^\bullet_{(a,i)})_{(a,i)\in m})} \quad \text{s.t.} \begin{cases} m \in \mathcal{M}_{\mathrm{f}}(|\mathcal{A}|) \\ \Gamma^{\bullet\prime} \uplus \biguplus_{(a,i)\in m} \Gamma^\bullet_{(a,i)} = \Gamma^\bullet \end{cases}$$

$$\frac{\Gamma^\bullet \vdash_\alpha M : n+1}{\Gamma^\bullet \vdash_\alpha M{-}1 : n}\ \text{pred} \qquad \frac{\Gamma^\bullet \vdash_\alpha M : n}{\Gamma^\bullet \vdash_\alpha M{+}1 : n+1}\ \text{succ}$$

$$\frac{\Gamma^\bullet \vdash_\beta M : 0 \qquad \Delta^\bullet \vdash_\alpha N : a}{\Gamma^\bullet \uplus \Delta^\bullet \vdash_{\beta\alpha} \mathtt{if}(M, N, P) : a}\ \text{if}_0 \qquad \frac{\Gamma^\bullet \vdash_\beta M : n+1 \qquad \Delta^\bullet \vdash_\alpha P : a}{\Gamma^\bullet \uplus \Delta^\bullet \vdash_{\beta\alpha} \mathtt{if}(M, N, P) : a}\ \text{if}_S \qquad \frac{\Gamma^\bullet \vdash_\alpha M : a}{\Gamma^\bullet \vdash_{\alpha X} X \cdot M : a}\ \text{par}$$

Weighted Judgements : $\pi :: x_1^{m_1}, \ldots, x_n^{m_n} \vdash_\alpha M : a$

Terms:

$$M, N, P ::= x \mid \lambda x^A.M \mid (M)\,N \mid \mathtt{fix}(M) \mid \underline{0} \mid M\mathtt{+1} \mid M\mathtt{-1} \mid \mathtt{if}(M, N, P) \mid \mathtt{Rand}$$

Types: $A, B, C ::= \mathtt{Int} \mid A \Rightarrow B$

$$x_1 : C_1, \ldots, x_n : C_n \vdash M : A$$

Web: $((m_1, \ldots, m_n)\,,\, a) \in \left(\mathcal{M}_f\,(|\mathcal{C}_1|) \times \cdots \times \mathcal{M}_f\,(|\mathcal{C}_n|)\right) \times |\mathcal{A}|$

Weight: $\omega(\pi) \acute{=} \alpha \in \overline{\mathbb{R}^+}$

Definition

Let M be a PPCF term such that $\Gamma \vdash M : A$.
The weighted relational semantics of M is given by:

$$[\![M]\!]_a^{\Gamma^\bullet} \acute{=} \sum_{\pi::\Gamma^\bullet \vdash M:a} \omega(\pi) \qquad \text{where} \begin{cases} \Gamma = x_1 : C_1, \ldots, x_n : C_n \\ \Gamma^\bullet = x_1^{C_1} : m_1, \ldots, x_n^{C_n} : m_n \\ a \in |A|\,,\ \text{and } m_i \in \mathcal{M}_f\,(|C_i|)\,,\ \forall i \in \{1, \ldots, n\}. \end{cases}$$

The sum is over the finite derivation trees π of $\Gamma^\bullet \vdash M : a$. The weight of π is $\omega(\pi)$.

Examples $\texttt{Once} \acute{=} \lambda x^{\texttt{Int}}.\texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42})$

$$\dfrac{\dfrac{m = [0]}{x : m \vdash_1 x : 0}\,\text{var} \qquad \dfrac{\dfrac{p = [2], k = 0,1}{\vdash_{\frac{1}{2}} \texttt{Rand} : (p,k)}\,\text{rand} \qquad \dfrac{}{\vdash_1 \underline{2} : 2}\,\text{nat}}{\vdash_{\frac{1}{2}} (\texttt{Rand})\,\underline{2} : k}\,\text{app}}{\dfrac{x : m \vdash_{\frac{1}{2}} \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : k}{\vdash_{\frac{1}{2}} \lambda x^{\texttt{Int}}.\texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : (m,k)}\,\text{abs}}\,\text{if}_0$$

$$\dfrac{\dfrac{\dfrac{m = [n+1]}{x : m \vdash_1 x : n+1}\,\text{var} \qquad \dfrac{k = 42}{\vdash_1 \underline{42} : k}\,\text{nat}}{x : m \vdash_1 \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : k}\,\text{if}_s}{\vdash_1 \lambda x^{\texttt{Int}}.\texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : (m,k)}\,\text{abs}$$

$$[\![\texttt{Once}]\!]_{(m,a)} = \sum_{\pi \vdash \texttt{Once}:(m,a)} \omega(\pi) = \begin{cases} ([0], 0) \mapsto \frac{1}{2} \\ ([0], 1) \mapsto \frac{1}{2} \\ ([n+1], 42) \mapsto 1 \text{ with } n \in \mathbb{N} \\ (m, a) \mapsto 0 \text{ otherwise.} \end{cases}$$

Examples $\texttt{Once} \overset{\prime}{=} \lambda x^{\texttt{Int}}.\texttt{if}(x, \tfrac{1}{2}\texttt{Coin}, \underline{42})$

$$\dfrac{\dfrac{m = [0]}{x : m \vdash_1 x : 0}\,\text{var} \qquad \dfrac{\dfrac{p = [2],\, k = 0, 1}{\vdash_{\frac{1}{2}} \texttt{Rand} : (p, k)}\,\text{rand} \qquad \dfrac{}{\vdash_1 \underline{2} : 2}\,\text{nat}}{\vdash_{\frac{1}{2}} (\texttt{Rand})\,\underline{2} : k}\,\text{app}}{\dfrac{x : m \vdash_{\frac{1}{2}} \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : k}{\vdash_{\frac{1}{2}} \lambda x^{\texttt{Int}}.\texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : (m, k)}\,\text{abs}}\,\text{if}_0$$

$$\dfrac{\dfrac{\dfrac{m = [n+1]}{x : m \vdash_1 x : n+1}\,\text{var} \qquad \dfrac{k = 42}{\vdash_1 \underline{42} : k}\,\text{nat}}{x : m \vdash_1 \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : k}\,\text{if}_s}{\vdash_1 \lambda x^{\texttt{Int}}.\texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : (m, k)}\,\text{abs}$$

$$[\![\texttt{Once}]\!]_{(m,a)} = \sum_{\pi \vdash \texttt{Once}:(m,a)} \omega(\pi) = \begin{cases} (\ [0]\ ,\ 0\ ) \mapsto \frac{1}{2} \\ (\ [0]\ ,\ 1\ ) \mapsto \frac{1}{2} \\ (\ [n+1], 42) \mapsto 1 \text{ with } n \in \mathbb{N} \\ (\ m\ ,\ a\ ) \mapsto 0 \text{ otherwise.} \end{cases}$$

Examples $\texttt{Once} \stackrel{\prime}{=} \lambda x^{\texttt{Int}}.\texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42})$

$$\dfrac{\dfrac{m = [0]}{x : m \vdash_1 x : 0}\,\text{var} \qquad \dfrac{\dfrac{p = [2], k = 0, 1}{\vdash_{\frac{1}{2}} \texttt{Rand} : (p, k)}\,\text{rand} \qquad \dfrac{}{\vdash_1 \underline{2} : 2}\,\text{nat}}{\vdash_{\frac{1}{2}} (\texttt{Rand})\,\underline{2} : k}\,\text{app}}{\dfrac{x : m \vdash_{\frac{1}{2}} \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : k}{\vdash_{\frac{1}{2}} \lambda x^{\texttt{Int}}.\texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : (m, k)}\,\text{abs}}\,\text{if}_0$$

$$\dfrac{\dfrac{\dfrac{m = [n+1]}{x : m \vdash_1 x : n+1}\,\text{var} \qquad \dfrac{k = 42}{\vdash_1 \underline{42} : k}\,\text{nat}}{x : m \vdash_1 \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : k}\,\text{if}_s}{\vdash_1 \lambda x^{\texttt{Int}}.\texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : (m, k)}\,\text{abs}$$

$$[\![\texttt{Once}]\!]_{(m,a)} = \sum_{\pi \vdash \texttt{Once}:(m,a)} \omega(\pi) = \begin{cases} ([0], 0) \mapsto \frac{1}{2} \\ ([0], 1) \mapsto \frac{1}{2} \\ ([n+1], 42) \mapsto 1 \text{ with } n \in \mathbb{N} \\ (m, a) \mapsto 0 \text{ otherwise.} \end{cases}$$

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$$\dfrac{\dfrac{\dfrac{m = [n+1]}{x : m \vdash_1 x : n+1}\,\text{var} \qquad \dfrac{k = 42}{\vdash_1 \underline{42} : k}\,\text{nat}}{x : m \vdash_1 \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : k}\,\text{if}_s}{\vdash_1 \lambda x^{\texttt{Int}}.\texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : (m, k)}\,\text{abs}$$

$$[\![\texttt{Once}]\!]_{(m,a)} = \sum_{\pi \vdash \texttt{Once}:(m,a)} \omega(\pi) = \begin{cases} ([0], 0) \mapsto \frac{1}{2} \\ ([0], 1) \mapsto \frac{1}{2} \\ ([n+1], 42) \mapsto 1 & \text{with } n \in \mathbb{N} \\ (m, a) \mapsto 0 & \text{otherwise.} \end{cases}$$

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$$\dfrac{\dfrac{\dfrac{m = [n+1]}{x : m \vdash_1 x : n+1}\,\text{var} \qquad \dfrac{k = 42}{\vdash_1 \underline{42} : k}\,\text{nat}}{x : m \vdash_1 \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : k}\,\text{if}_s}{\vdash_1 \lambda x^{\texttt{Int}}.\texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : (m, k)}\,\text{abs}$$

$$[\![\texttt{Once}]\!]_{(m,a)} = \sum_{\pi \vdash \texttt{Once}:(m,a)} \omega(\pi) = \begin{cases} ([0], 0) \mapsto \frac{1}{2} \\ ([0], 1) \mapsto \frac{1}{2} \\ ([n+1], 42) \mapsto 1 & \text{with } n \in \mathbb{N} \\ (m, a) \mapsto 0 & \text{otherwise.} \end{cases}$$

$$\dfrac{\dfrac{\pi}{\vdash_{[\![\texttt{Once}]\!]_{(m,k)}} \texttt{Once} : (m,k)} \qquad \dfrac{\tau}{\Gamma^\bullet \vdash_{[\![P]\!]_a} P : a} \; \forall (a,i) \in m}{\Gamma^\bullet \vdash_{[\![\texttt{Once}]\!]_{(m,k)} \prod_{a \in m} [\![P]\!]_a} (\texttt{Once})\, P : k} \text{app}$$

Composition Formula:

$$[\![(\texttt{Once})P]\!]_k \quad = \quad \sum_{m \in \mathcal{M}_f(\mathbb{N})} [\![\texttt{Once}]\!]_{(m,k)} \prod_{(a,i)\in m} [\![P]\!]_a^{\Gamma^\bullet}$$

Computation:

$$[\![\texttt{Once}]\!] = \begin{cases} ([0],\ 0\ ) \mapsto \frac{1}{2} \\ ([0],\ 1\ ) \mapsto \frac{1}{2} \\ ([a], 42) \mapsto 1 \text{ if } a \neq 0 \\ (\ m,\ \ a\ ) \mapsto 0 \text{ otherwise.} \end{cases}$$

Power Series:

$$\begin{aligned} [\![(\texttt{Once})\, P]\!]_0 &= [\![(\texttt{Once})\, P]\!]_1 = \tfrac{1}{2}\, [\![P]\!]_0 \\ [\![(\texttt{Once})\, P]\!]_{42} &= \sum_{a \geq 1} [\![P]\!]_a \\ [\![(\texttt{Once})\, P]\!]_a &= 0 \text{ otherwise} \end{aligned}$$

Examples $\texttt{Twice} \stackrel{\prime}{=} \lambda x^{\texttt{Int}}.\texttt{if}(x, \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}), \texttt{if}(x, \underline{42}, \underline{0}))$

$$\dfrac{\dfrac{\dfrac{m = [0]}{x : m \vdash_1 x : 0}\ \text{var} \qquad \dfrac{\dfrac{p = [0]}{x : p \vdash_1 x : 0} \qquad \dfrac{k = 0, 1}{\vdash_{\frac{1}{2}} \frac{1}{2}\texttt{Coin} : k}}{x : p \vdash_{\frac{1}{2}} \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : k}\ \text{if}_0}{x : m + p \vdash_{\frac{1}{2}} \texttt{if}(x, \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}), \texttt{if}(x, \underline{42}, \underline{0})) : k}\ \text{if}_0}{\vdash_{\frac{1}{2}} \lambda x^{\texttt{Int}}.\texttt{if}(x, \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}), \texttt{if}(x, \underline{42}, \underline{0})) : (m + p, k)}\ \text{abs}$$

$$[\![\texttt{Twice}]\!] \stackrel{\prime}{=} \begin{cases} (\,[0,0],\ 0\,) \mapsto \frac{1}{2} \\ (\,[0,0],\ 1\,) \mapsto \frac{1}{2} \\ (\,[0,a], 42) \mapsto 1 + 1 & \text{if } a \neq 0 \\ (\,[a,b],\ 0\,) \mapsto 1 & \text{if } a \neq 0, b \neq 0 \\ (\ m\ ,\ a\,) \mapsto 0 & \text{otherwise.} \end{cases}$$

Examples $\quad \texttt{Twice} \mathrel{\acute{=}} \lambda x^{\texttt{Int}}.\texttt{if}(x, \texttt{if}(x, \tfrac{1}{2}\texttt{Coin}, \underline{42}), \texttt{if}(x, \underline{42}, \underline{0}))$

$$\dfrac{\dfrac{m = [0]}{x : m \vdash_1 x : 0}\,\text{var} \qquad \dfrac{\dfrac{p = [0]}{x : p \vdash_1 x : 0} \qquad \dfrac{k = 0, 1}{\vdash_{\frac{1}{2}} \tfrac{1}{2}\texttt{Coin} : k}}{x : p \vdash_{\frac{1}{2}} \texttt{if}(x, \tfrac{1}{2}\texttt{Coin}, \underline{42}) : k}\,\text{if}_0}{\dfrac{x : m + p \vdash_{\frac{1}{2}} \texttt{if}(x, \texttt{if}(x, \tfrac{1}{2}\texttt{Coin}, \underline{42}), \texttt{if}(x, \underline{42}, \underline{0})) : k}{\vdash_{\frac{1}{2}} \lambda x^{\texttt{Int}}.\texttt{if}(x, \texttt{if}(x, \tfrac{1}{2}\texttt{Coin}, \underline{42}), \texttt{if}(x, \underline{42}, \underline{0})) : (m + p, k)}\,\text{abs}}\,\text{if}_0$$

$$[\![\texttt{Twice}]\!] \mathrel{\acute{=}} \begin{cases} (\,[0,0],\ 0\,) \mapsto \frac{1}{2} \\ (\,[0,0],\ 1\,) \mapsto \frac{1}{2} \\ (\,[0,a], 42) \mapsto 1 + 1 & \text{if } a \neq 0 \\ (\,[a,b],\ 0\,) \mapsto 1 & \text{if } a \neq 0, b \neq 0 \\ (\ m\ ,\ a\,) \mapsto 0 & \text{otherwise.} \end{cases}$$

Examples $\texttt{Twice} \acute{=} \lambda x^{\texttt{Int}}.\texttt{if}(x, \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}), \texttt{if}(x, \underline{42}, \underline{0}))$

$$\dfrac{\dfrac{m = [0]}{x : m \vdash_1 x : 0}\,\text{var} \qquad \dfrac{\dfrac{p = [n+1]}{x : p \vdash_1 x : n+1} \qquad \dfrac{}{\vdash_1 \underline{42} : 42}}{x : p \vdash_{\frac{1}{2}} \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}) : 42}\,\text{if}_s}{\dfrac{x : m + p \vdash_1 \texttt{if}(x, \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}), \texttt{if}(x, \underline{42}, \underline{0})) : 42}{\vdash_1 \lambda x^{\texttt{Int}}.\texttt{if}(x, \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}), \texttt{if}(x, \underline{42}, \underline{0})) : (m + p, 42)}\,\text{abs}}\,\text{if}_0$$

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$$\dfrac{\dfrac{\dfrac{m = [n+1]}{x : m \vdash_1 x : n+1} \qquad \dfrac{\dfrac{p = [0]}{x : p \vdash_1 x : 0} \qquad \dfrac{}{\vdash_1 \underline{42} : 42}}{x : p \vdash_{\frac{1}{2}} \texttt{if}(x, \underline{42}, \underline{0}) : 42}\,\text{if}_s}{x : m + p \vdash_1 \texttt{if}(x, \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}), \texttt{if}(x, \underline{42}, \underline{0})) : 42}\,\text{if}_s}{\vdash_1 \lambda x^{\texttt{Int}}.\texttt{if}(x, \texttt{if}(x, \frac{1}{2}\texttt{Coin}, \underline{42}), \texttt{if}(x, \underline{42}, \underline{0})) : (m+p, 42)}\,\text{abs}$$

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$$\dfrac{\dfrac{\pi}{\vdash_\alpha \texttt{Twice} : (m,k)} \qquad \dfrac{\tau}{\vdash_{[\![P]\!]_a} x : a}\ \forall (a,i) \in m}{\vdash_{[\![\texttt{Twice}]\!]_{(m,k)} \prod_{a\in m}[\![P]\!]_a} (\texttt{Twice})\, x : k}\ \text{app}$$

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$$[\![(\texttt{Twice})P]\!]_k \quad = \quad \sum_{m \in \mathcal{M}_f(\mathbb{N})} [\![\texttt{Twice}]\!]_{(m,k)} \prod_{(a,i)\in m} [\![P]\!]_a$$

Computation:

$$[\![\texttt{Twice}]\!] \begin{cases} ([0,0],\ 0\ ) \mapsto \frac{1}{2} \\ ([0,0],\ 1\ ) \mapsto \frac{1}{2} \\ ([0,a], 42) \mapsto 2 \text{ if } a \neq 0 \\ ([a,b],\ 0\ ) \mapsto 1 \text{ if } a \neq 0, b \neq 0 \\ (\ m\ ,\ a\ ) \mapsto 0 \text{ otherwise.} \end{cases}$$

Power Series:

$$[\![(\texttt{Twice})\, x]\!]_0 \quad = \quad \frac{1}{2}\, [\![x]\!]_0^2 + \sum_{a,b \geq 1} [\![x]\!]_a\, [\![x]\!]_b$$

$$[\![(\texttt{Twice})\, x]\!]_1 \quad = \quad \frac{1}{2}\, [\![x]\!]_0^2$$

$$[\![(\texttt{Twice})\, x]\!]_{42} \quad = \quad 2 \sum_{a \geq 1} [\![x]\!]_0\, [\![x]\!]_a$$

Examples $\texttt{choose}() \stackrel{\prime}{=} \texttt{fix}(\lambda x^A.x)$

$$\dfrac{\dfrac{\dfrac{m = [a]}{x : m \vdash_1 x : a}\,\text{var}}{\vdash_1 \lambda x^A.x : (m, x)}\,\text{abs} \qquad \dfrac{\vdots}{\vdash_\alpha \texttt{fix}(\lambda x^A.x) : a}\,\forall (a,i) \in m}{\vdash_\alpha \texttt{fix}(\lambda x^A.x) : a}\,\text{fix}$$

Definition

Let M be a PPCF term such that $\Gamma \vdash M : A$. The weighted relational semantics is :

$$[\![M]\!]_a^{\Gamma^\bullet} \stackrel{\prime}{=} \sum_{\pi :: \Gamma^\bullet \vdash M : a} \omega(\pi)$$

The sum is over the finite derivation trees π of $\Gamma^\bullet \vdash M : a$.

Theorem (Adequacy)

Let $M : \texttt{nat}$ be a closed program. Then for all n, $\textbf{Proba}(M \xrightarrow{*} \underline{n}) = [\![M]\!]_n$.

Conclusion: $[\![\texttt{fix}(\lambda x^A.x)]\!] = 0$

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Conclusion: $\left[\!\left[\mathtt{fix}(\lambda x^A.x)\right]\!\right] = 0$

Examples (waittt) $\frac{1}{2}\texttt{Coin} = \texttt{fix}(\lambda x^{\texttt{Int}}.\texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0}))$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_{\frac{1}{2}} \frac{1}{2}\texttt{Coin} : 0} \qquad \dfrac{m = [a]}{x : m \vdash_1 x : a}\,\text{var}}{x : m \vdash_{\frac{1}{2}} \texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0}) : a}\,\text{if}_0}{\vdash_{\frac{1}{2}} \lambda x^{\texttt{Int}}.\texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0}) : (m, a)}\,\text{abs} \qquad \dfrac{\pi}{\vdash_{\alpha_i} \texttt{fix}(\lambda x^{\texttt{Int}}.\texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0})) : a}\,(a,i) \in m}{\vdash_{\frac{1}{2}\prod_i \alpha_i} \texttt{fix}(\lambda x^{\texttt{Int}}.\texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0})) : a}\,\text{fix}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_{\frac{1}{2}} \frac{1}{2}\texttt{Coin} : 1} \qquad \dfrac{m = [\,]}{x : m \vdash_1 \underline{0} : 0}\,\text{nat}}{x : m \vdash_{\frac{1}{2}} \texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0}) : 0}\,\text{if}_s}{\vdash_{\frac{1}{2}} \lambda x^{\texttt{Int}}.\texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0}) : (m, 0)}\,\text{abs} \qquad \dfrac{\pi}{\vdash_{\alpha_i} \texttt{fix}(\lambda x^{\texttt{Int}}.\texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0})) : a}\,(a,i) \in m}{\vdash_{\frac{1}{2}\prod_i \alpha_i} \texttt{fix}(\lambda x^{\texttt{Int}}.\texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0})) : 0}\,\text{fix}$$

(waittt) $\frac{1}{2}\texttt{Coin} = \texttt{fix}(\lambda x^{\texttt{Int}}.\texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0}))$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_{\frac{1}{2}} \frac{1}{2}\texttt{Coin} : 0} \qquad \dfrac{m = [a]}{x : m \vdash_1 x : a}\ \text{var}}{x : m \vdash_{\frac{1}{2}} \texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0}) : a}\ \text{if}_0}{\vdash_{\frac{1}{2}} \lambda x^{\texttt{Int}}.\texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0}) : (m, a)}\ \text{abs} \qquad \dfrac{\pi}{\vdash_{\alpha_i} \texttt{fix}(\lambda x^{\texttt{Int}}.\texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0})) : a}\ (a,i) \in m}{\vdash_{\frac{1}{2}\prod_i \alpha_i} \texttt{fix}(\lambda x^{\texttt{Int}}.\texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0})) : a}\ \text{fix}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_{\frac{1}{2}} \frac{1}{2}\texttt{Coin} : 1} \qquad \dfrac{m = [\,]}{x : m \vdash_1 \underline{0} : 0}\ \text{nat}}{x : m \vdash_{\frac{1}{2}} \texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0}) : 0}\ \text{if}_s}{\vdash_{\frac{1}{2}} \lambda x^{\texttt{Int}}.\texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0}) : (m, 0)}\ \text{abs} \qquad \dfrac{\pi}{\vdash_{\alpha_i} \texttt{fix}(\lambda x^{\texttt{Int}}.\texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0})) : a}\ (a,i) \in m}{\vdash_{\frac{1}{2}\prod_i \alpha_i} \texttt{fix}(\lambda x^{\texttt{Int}}.\texttt{if}(\frac{1}{2}\texttt{Coin}, x, \underline{0})) : 0}\ \text{fix}$$

Examples

$$(\text{waittt})\ \tfrac{1}{2}\texttt{Coin} = \texttt{fix}(\lambda x^{\texttt{Int}}.\texttt{if}(\tfrac{1}{2}\texttt{Coin}, x, \underline{0}))$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_{\frac{1}{2}} \tfrac{1}{2}\texttt{Coin} : 0} \quad \dfrac{m = [a]}{x : m \vdash_1 x : a}\ \text{var}}{x : m \vdash_{\frac{1}{2}} \texttt{if}(\tfrac{1}{2}\texttt{Coin}, x, \underline{0}) : a}\ \text{if}_0}{\vdash_{\frac{1}{2}} \lambda x^{\texttt{Int}}.\texttt{if}(\tfrac{1}{2}\texttt{Coin}, x, \underline{0}) : (m, a)}\ \text{abs} \quad \dfrac{\pi}{\vdash_{\alpha_i} \texttt{fix}(\lambda x^{\texttt{Int}}.\texttt{if}(\tfrac{1}{2}\texttt{Coin}, x, \underline{0})) : a}\ (a, i) \in m}{\vdash_{\frac{1}{2}\prod_i \alpha_i} \texttt{fix}(\lambda x^{\texttt{Int}}.\texttt{if}(\tfrac{1}{2}\texttt{Coin}, x, \underline{0})) : a}\ \text{fix}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash_{\frac{1}{2}} \tfrac{1}{2}\texttt{Coin} : 1} \quad \dfrac{m = [\,]}{x : m \vdash_1 \underline{0} : 0}\ \text{nat}}{x : m \vdash_{\frac{1}{2}} \texttt{if}(\tfrac{1}{2}\texttt{Coin}, x, \underline{0}) : 0}\ \text{if}_s}{\vdash_{\frac{1}{2}} \lambda x^{\texttt{Int}}.\texttt{if}(\tfrac{1}{2}\texttt{Coin}, x, \underline{0}) : (m, 0)}\ \text{abs} \quad \dfrac{\cancel{\pi}}{\cancel{\vdash_{\alpha_i} \texttt{fix}(\lambda x^{\texttt{Int}}.\texttt{if}(\tfrac{1}{2}\texttt{Coin}, x, \underline{0})) : a}}\ \cancel{(a, i) \in m}}{\vdash_{\frac{1}{2}\cancel{\prod_i \alpha_i}} \texttt{fix}(\lambda x^{\texttt{Int}}.\texttt{if}(\tfrac{1}{2}\texttt{Coin}, x, \underline{0})) : 0}\ \text{fix}$$

Examples (waittt) $\frac{1}{2}\mathtt{Coin} = \mathtt{fix}(\lambda x^{\mathtt{Int}}.\mathtt{if}(\frac{1}{2}\mathtt{Coin}, x, \underline{0}))$

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\dfrac{}{\vdash_{\frac{1}{2}} \frac{1}{2}\mathtt{Coin} : 0}
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}{x : [0] \vdash_{\frac{1}{2}} \mathtt{if}(\frac{1}{2}\mathtt{Coin}, x, \underline{0}) : 0}\,\text{if}_0
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\dfrac{}{\vdash_{\frac{1}{2}} \frac{1}{2}\mathtt{Coin} : 1}
\qquad
\dfrac{}{x : [\,] \vdash_{1} \underline{0} : 0}\,\text{nat}
}{x : [\,] \vdash_{\frac{1}{2}} \mathtt{if}(\frac{1}{2}\mathtt{Coin}, x, \underline{0}) : 0}\,\text{if}_s
}{\vdash_{\frac{1}{2}} \lambda x^{\mathtt{Int}}.\mathtt{if}(\frac{1}{2}\mathtt{Coin}, x, \underline{0}) : ([\,], 0)}\,\text{abs}
}{\vdash_{\frac{1}{2}} \mathtt{fix}(\lambda x^{\mathtt{Int}}.\mathtt{if}(\frac{1}{2}\mathtt{Coin}, x, \underline{0})) : 0}\,\text{fix}
}{
\dfrac{\cdot^{\cdot^{\cdot^{\,k}}}}{\vdash_{\frac{1}{2^k}} \mathtt{fix}(\lambda x^{\mathtt{Int}}.\mathtt{if}(\frac{1}{2}\mathtt{Coin}, x, \underline{0})) : 0}
}
}{\vdash_{\frac{1}{2^{k+1}}} \mathtt{fix}(\lambda x^{\mathtt{Int}}.\mathtt{if}(\frac{1}{2}\mathtt{Coin}, x, \underline{0})) : 0}\,\text{fix}
$$

$$\left[\!\left[\mathtt{fix}(\lambda x^{\mathtt{Int}}.\mathtt{if}(\tfrac{1}{2}\mathtt{Coin}, x, \underline{0}))\right]\!\right]_0 = \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} = 1$$

Examples

$$\texttt{choose}(M_1,\ldots,M_{n+1}) \acute{=} \texttt{if}((\texttt{rand})\,\underline{n+1}, M_{n+1}, \texttt{choose}(M_1,\ldots,M_n))$$

$$\dfrac{\dfrac{}{\vdash_{\frac{1}{n+1}} (\texttt{rand})\,\underline{n+1} : 0} \qquad \dfrac{}{\vdash_{[\![M_{n+1}]\!]_a} M_{n+1} : a}}{\vdash_{\frac{1}{n+1}[\![M_{n+1}]\!]_a} \texttt{if}((\texttt{rand})\,\underline{n+1}, M_{n+1}, \texttt{choose}(M_1,\ldots,M_n)) : a}\ \text{if}_0$$

$$\dfrac{\dfrac{1 \le k \le n}{\vdash_{\frac{1}{n+1}} (\texttt{rand})\,\underline{n+1} : k} \qquad \dfrac{1 \le j \le n}{\vdash_{\frac{1}{n}[\![M_j]\!]_a} \texttt{choose}(M_1,\ldots,M_n) : a}}{\vdash_{\frac{1}{n+1}\frac{1}{n}[\![M_j]\!]_a} \texttt{if}((\texttt{rand})\,\underline{n+1}, M_{n+1}, \texttt{choose}(M_1,\ldots,M_n)) : a}\ \text{if}_s$$

$$\begin{aligned}[\![\texttt{choose}(M_1,\ldots,M_{n+1})]\!]_a &= \tfrac{1}{n+1}[\![M_{n+1}]\!]_a + \tfrac{1}{n+1}\sum_{k=1}^{n}\sum_{j=1}^{n}\tfrac{1}{n}[\![M_j]\!]_a \\ &= \tfrac{1}{n+1}\sum_{j=1}^{n+1}[\![M_j]\!]_a\end{aligned}$$

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$\texttt{choose}(M_1, \ldots, M_{n+1}) \acute{=} \texttt{if}((\texttt{rand})\,\underline{n+1}, M_{n+1}, \texttt{choose}(M_1, \ldots, M_n))$

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Examples

$$\texttt{choose}(M_1, \ldots, M_{n+1}) \acute{=} \texttt{if}((\texttt{rand})\, \underline{n+1}, M_{n+1}, \texttt{choose}(M_1, \ldots, M_n))$$

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$$\begin{aligned} [\![\texttt{choose}(M_1, \ldots, M_{n+1})]\!]_a &= \tfrac{1}{n+1} [\![M_{n+1}]\!]_a + \tfrac{1}{n+1} \sum_{k=1}^{n} \sum_{j=1}^{n} \tfrac{1}{n} [\![M_j]\!]_a \\ &= \tfrac{1}{n+1} \sum_{j=1}^{n+1} [\![M_j]\!]_a \end{aligned}$$

Break

- $\overline{\mathbb{R}^+}$-weighted semantics for Probabilistic PCF.
- Intersection type system for computing semantics

D. de Carvalho. Execution Time of λ-terms via denotational Semantics and Intersection Types. *Preprint, 2009.*

T. Ehrhard, M. Pagani, C. Tasson. Probabilisitc Coherence Spaces are Fully Abstract for Probabilistic PCF. Preprint, 2013.

- Interpretation of terms as power series (Quantitative semantics)

J.-Y. Girard. Normal Functor Power Series. *Ann. Pure Appl. Logic, 1988.*

T. Ehrhard. Finiteness Spaces. *Math. Struct. Comput. Sci., 2005.*

Next Part:

- Proof of Full Abstraction
- Topology and Quantitive Semantics, some examples

Towards Full Abstraction, testing terms with parameters:

$$\forall a \in |A|, \quad \mathcal{P}(a) : \ A \Rightarrow \texttt{Int}, \quad \mathcal{N}(a) : \ A.$$

If $a \in |\texttt{Int}|$ then $a = n$ and

$$\mathcal{P}(n) = \lambda x^{\texttt{Int}}.\texttt{if}(x = \underline{n}, \underline{0}, \Omega_{\texttt{Int}})$$
$$\mathcal{N}(n) = \underline{n}$$

If $a \in |B \Rightarrow C|$ then $a = ([b_1, \ldots, b_n], c)$ and

$$\mathcal{P}(a) = \lambda z^{B \Rightarrow C}.(\mathcal{P}(c))\left((z)\,\texttt{choose}(X_i \cdot \mathcal{N}(b_i))_{i=1}^{n}\right)$$
$$\mathcal{N}(a) = \lambda x^{B}.\texttt{if}(\wedge_{i=1}^{n}\,(\mathcal{P}(b_i))\,x, \mathcal{N}(c), \Omega_C)$$

with

$$\texttt{if}(\wedge_{i=1}^{0} M_i, N, P) \acute{=} N \qquad \texttt{if}(\wedge_{i=1}^{n+1} M_i, N, P) \acute{=} \texttt{if}(M_{n+1}, \texttt{if}(\wedge_{i=1}^{n} M_i, N, P), P)$$
$$\texttt{if}(M = \underline{0}, N, P) \acute{=} \texttt{if}(M, N, P) \qquad \texttt{if}(M = \underline{n+1}, N, P) \acute{=} \texttt{if}(M-1 = \underline{n}, N, P)$$

Theorem: For all $a \in |A|$, $[\![\mathcal{P}(a)]\!]$ and $[\![\mathcal{N}(a)]\!]$ are **power series** in their parameters. Let $\texttt{sk}(a)$ be the monomial having of these parameters occuring with degree one. The coefficient of $\texttt{sk}(a)$ in:

$$[\![\mathcal{P}(a)]\!]_{(m,0)} \text{ is non zero } \iff m = [a]$$
$$[\![\mathcal{N}(a)]\!]_{a'} \text{ is non zero } \iff a' = a$$

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with $\texttt{if}(\wedge_{i=1}^{0} M_i, N, P) \acute{=} N$ $\quad \texttt{if}(\wedge_{i=1}^{n+1} M_i, N, P) \acute{=} \texttt{if}(M_{n+1}, \texttt{if}(\wedge_{i=1}^{n} M_i, N, P), P)$

$\texttt{if}(M = \underline{0}, N, P) \acute{=} \texttt{if}(M, N, P)$ $\quad \texttt{if}(M = \underline{n+1}, N, P) \acute{=} \texttt{if}(M - 1 = \underline{n}, N, P)$

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Case $\texttt{Int}$: Compute coefficient of $\mathtt{sk}(n) = 1$ in $[\![\mathcal{P}(n)]\!]$

with $\mathcal{P}(n) = \lambda x^{\mathtt{Int}}.\mathtt{if}(x = n, \underline{0}, \Omega_{\mathtt{Int}})$

The only possible proof tree:

$$\dfrac{\dfrac{\dfrac{m = [n]}{x : m \vdash_1 x : n}\,\text{var} \qquad \dfrac{}{\vdash_1 \underline{0} : 0}\,\text{nat}}{x : m \vdash_1 \mathtt{if}(x = \underline{n}, 0, \Omega_{\mathtt{Int}}) : 0}\,\text{if}_0}{\vdash_1 \lambda x^{\mathtt{Int}}.\mathtt{if}(x = \underline{n}, 0, \Omega_{\mathtt{Int}}) : (m, 0)}\,\text{abs}$$

Conclusion:

The coefficient of $\mathtt{sk}(a)$ in $[\![\mathcal{P}(n)]\!]_{(m,0)}$ is non zero iff $m = [n]$.

Case $\texttt{Int}$: Compute coefficient of $\texttt{sk}(n) = 1$ in $[\![\mathcal{P}(n)]\!]$

with $\mathcal{P}(n) = \lambda x^{\texttt{Int}}.\texttt{if}(x = n, \underline{0}, \Omega_{\texttt{Int}})$

The only possible proof tree:

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Conclusion:

The coefficient of $\texttt{sk}(a)$ in $[\![\mathcal{P}(n)]\!]_{(m,0)}$ is non zero iff $m = [n]$.

Case $B \Rightarrow C$: Compute coefficient of $\mathtt{sk}(a) = \prod_{i=1}^{n} X_i$ in $[\![\mathcal{P}(a)]\!]$

$a = ([b_1, \dots, b_n], c)$ with $\mathcal{P}(a) = \lambda z^{B \Rightarrow C}.(\mathcal{P}(c))\,((z)\,\mathtt{choose}(X_i \cdot \mathcal{N}(b_i))_{i=1}^{n})$

Induction Hypothesis $\mathtt{IH}(b)$:

The coefficient of $\mathtt{sk}(b)$ in:

$$[\![\mathcal{P}(b)]\!]_{(m,0)} \text{ is non zero } \iff m = [b]$$

$$[\![\mathcal{N}(b)]\!]_{b'} \text{ is non zero } \iff b' = b$$

Possible proof tree:

$$\dfrac{\mathtt{IH}(B)\text{: } b_{i_j} = b'_j}{\vdash_{\beta_{i_j}} \mathcal{N}(b_{i_j}) : b'_j}$$

$$\dfrac{\vdash_{\beta_{i_j}} \mathcal{N}(b_{i_j}) : b'_j}{\vdash_{x_{i_j}\beta_{i_j}} X_{i_j} \cdot \mathcal{N}(b_{i_j}) : b'_j}$$

$$\dfrac{1 \le j \le k' \qquad \vdash_{x_{i_j}\beta_{i_j}} X_{i_j} \cdot \mathcal{N}(b_{i_j}) : b'_j}{\vdash_{\frac{1}{k} x_{i_j}\beta_{i_j}} \mathtt{choose}(X_i \cdot \mathcal{N}(b_i))_{i=1}^{k} : b'_j}$$

$$\dfrac{\begin{array}{c} m = [(p', c)] \\ p' = [b'_1, \dots, b'_{k'}] \end{array}}{z : m \vdash_1 z : (p', c)}$$

$$\dfrac{z : m \vdash_1 z : (p', c) \qquad \vdash_{\frac{1}{k} x_{i_j}\beta_{i_j}} \mathtt{choose}(X_i \cdot \mathcal{N}(b_i))_{i=1}^{k} : b'_j}{z : m \vdash_{\beta} (z)\,\mathtt{choose}(X_i \cdot \mathcal{N}(b_i))_{i=1}^{k} : c}\ \text{app}$$

$$\dfrac{\mathtt{IH}(C)\text{: } q = [c]}{\vdash_{\gamma} \mathcal{P}(c) : (q, 0)}$$

$$\dfrac{\vdash_{\gamma} \mathcal{P}(c) : (q, 0) \qquad z : m \vdash_{\beta} (z)\,\mathtt{choose}(X_i \cdot \mathcal{N}(b_i))_{i=1}^{k} : c}{z : m \vdash_{\alpha\mathtt{sk}(a)} (\mathcal{P}(c))\,(z)\,\mathtt{choose}(X_i \cdot \mathcal{N}(b_i))_{i=1}^{k} : 0}\ \text{app}$$

$$\dfrac{z : m \vdash_{\alpha\mathtt{sk}(a)} (\mathcal{P}(c))\,(z)\,\mathtt{choose}(X_i \cdot \mathcal{N}(b_i))_{i=1}^{k} : 0}{\vdash_{\alpha\mathtt{sk}(a)} \lambda z^{B \Rightarrow C}.(\mathcal{P}(c))\,(z)\,\mathtt{choose}(X_i \cdot \mathcal{N}(b_i))_{i=1}^{k} : (m, 0)}\ \text{abs}$$

Conclusion: The coefficient of $\mathtt{sk}(a)$ in $[\![\mathcal{P}(a)]\!]_{(m,0)}$ is non zero iff $m = [([b_1, \dots, b_n], c)]$

$$[\![\mathcal{P}(a)]\!]_{([a],0)} = \sum_{\pi \vdash \mathcal{P}(a):([a],0)} \omega(\pi)$$

Case $B \Rightarrow C$: Compute coefficient of $\mathtt{sk}(a) = \prod_{i=1}^{n} X_i$ in $[\![\mathcal{P}(a)]\!]$

$a = ([b_1, \dots, b_n], c)$ with $\mathcal{P}(a) = \lambda z^{B \Rightarrow C}.(\mathcal{P}(c))\,((z)\,\mathtt{choose}(X_i \cdot \mathcal{N}(b_i))_{i=1}^{n})$

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$$\dfrac{\dfrac{\mathtt{IH}(C)\colon q = [c]}{\vdash_{\gamma} \mathcal{P}(c) : (q,0)} \qquad \dfrac{\dfrac{\begin{array}{c} m = [(p',c)] \\ p' = [b'_1, \dots, b'_{k'}] \end{array}}{z : m \vdash_1 z : (p',c)} \qquad \dfrac{1 \le j \le k' \qquad \dfrac{\dfrac{\mathtt{IH}(B)\colon b_{i_j} = b'_j}{\vdash_{\beta_{i_j}} \mathcal{N}(b_{i_j}) : b'_j}}{\vdash_{X_{i_j}\beta_{i_j}} X_{i_j} \cdot \mathcal{N}(b_{i_j}) : b'_j}}{\vdash_{\frac{1}{k} X_{i_j}\beta_{i_j}} \mathtt{choose}(X_i \cdot \mathcal{N}(b_i))_{i=1}^{k} : b'_j}}{z : m \vdash_{\beta} (z)\,\mathtt{choose}(X_i \cdot \mathcal{N}(b_i))_{i=1}^{k} : c}\ \text{app}}{\dfrac{z : m \vdash_{\alpha \mathtt{sk}(a)} (\mathcal{P}(c))\,(z)\,\mathtt{choose}(X_i \cdot \mathcal{N}(b_i))_{i=1}^{k} : 0}{\vdash_{\alpha \mathtt{sk}(a)} \lambda z^{B \Rightarrow C}.(\mathcal{P}(c))\,(z)\,\mathtt{choose}(X_i \cdot \mathcal{N}(b_i))_{i=1}^{k} : (m,0)}\ \text{abs}}\ \text{app}$$

Conclusion: The coefficient of $\mathtt{sk}(a)$ in $[\![\mathcal{P}(a)]\!]_{(m,0)}$ is non zero iff $m = [([b_1, \dots, b_n], c)]$

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Let $\mathtt{sk}(a)$ be the monomial having of these parameters occuring with degree one.
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$$[\![\mathcal{N}(a)]\!]_{a'} \text{ is non zero} \iff a' = a$$

Composition Formula:

$$[\![(\mathcal{P}(a))\, M]\!]_0 \quad = \quad \sum_m [\![\mathcal{P}(a)]\!]_{(m,0)} \prod_{b \in m} [\![M]\!]_b^{m(b)}$$

If M has no parameter, then

The coefficient of $\mathtt{sk}(a)$ in $[\![(\mathcal{P}(a))\, M]\!]_0$ is: $[\![\mathcal{P}(a)]\!]_{([a],0)}\, [\![M]\!]_a$.

These testing terms and monomial extract the ath coefficient of $[\![M]\!]$.

Tutorial on Quantitative Semantics (III)

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Logoi Summer School, Torino 2013

Road Map

1. An Application of Quantitative semantics:
   - Full Abstraction
2. Topological features
   - Probabilistic coherence spaces
   - Finiteness spaces
   - Convenient vector spaces

Full Abstraction:

A Bridge between Syntax and Semantics.

"*Full Abstraction studies connections between denotational and operational semantics.*" **LCF Considered as a Programming Language, Plotkin (77)**

Full Abstraction in a nutshell

FA relates Semantical and Observational equivalences:

$$[\![P]\!] = [\![Q]\!] \quad \underset{\text{Fullness}}{\overset{\text{Abstraction}}{\rightleftarrows}} \quad P \simeq_o Q$$

$$(\forall C[\,],\ C[P] \to^* v \iff C[Q] \to^* v)$$

How to prove Fullness:

1. By **contradiction**, start with $[\![P]\!] \neq [\![Q]\!]$
2. Find **testing function**: f such that $f\,[\![P]\!] \neq f\,[\![Q]\!]$
3. Prove **definability**:
   $\exists C[\cdot],\ \forall P,\ f\,[\![P]\!] = [\![C[P]]\!]$ and $C[P] \to^* p$.
4. Conclude:
   $\exists C[\cdot],\ [\![C[P]]\!] \neq [\![C[Q]]\!] \Rightarrow p \neq q \Rightarrow P \not\simeq_o Q$.

Full Abstraction in a nutshell

Game Semantics vs. PCF:

- Full abstraction
  - S. Abramsky, R. Jagadesen, and P. Malacaria. Full abstraction for PCF. *Information and Computation, 2000.*
  - *M. Hyland, and L. Ong. On full abstraction for PCF. Information and Computiation, 2000.*
- Definability result:
  any game corresponds to a Böhm tree (a PCF term in a canonical form).
- A quotient of game semantics: $f \equiv g \iff \forall h,\ hf = hg$
  forces semantical equivalence to correspond to observational equivalence

Probabilistic Setting:

- **Probabilistic Games** are fully abstract for Probabilistic PCF + References.
  - V. Danos, R. Harmer. Probabilistic game semantics. *ACM Transactions on Computational Logic, 2002.*
- $\mathfrak{R}$ **weighted relational semantics**
  - T. Ehrhard, M. Pagani, C. Tasson. Probabilisitc Coherence Spaces are Fully Abstract for Probabilistic PCF. *Preprint, 2013.*

Probabilistic Full Abstraction: $P, Q : A$

Semantical Equivalence: $[\![P]\!] = [\![Q]\!]$ with $[\![P]\!] : |\mathcal{A}| \to \mathbb{R}^+ \cup \infty$

$$\forall a \in |\mathcal{A}|,\ [\![P]\!]_a = [\![Q]\!]_a$$

Observational Equivalence: $P \simeq_o Q$ $\quad C[P] \Downarrow^\kappa v \iff C[Q] \Downarrow^\kappa v$

$$\forall n \in \mathbb{N},\ \forall C : A \Rightarrow \mathtt{Int},\ \mathbf{Proba}((C)\,P \xrightarrow{*} n) = \mathbf{Proba}((C)\,Q \xrightarrow{*} n)$$

Example:

$$\mathtt{fix}(\lambda x^{\mathtt{Int}}.\mathtt{if}(\tfrac{1}{2}\mathtt{Coin}, x, \underline{0})) \not\simeq_o \mathtt{fix}(\lambda x^{\mathtt{Int}}.\mathtt{if}((\mathtt{Rand})\,\underline{3}, x, \underline{0})).$$

Application of Adequacy Theorem: $\mathbf{Proba}(P \xrightarrow{*} \underline{0}) = [\![P]\!]_0$.

$$\left[\!\left[\mathtt{fix}(\lambda x^{\mathtt{Int}}.\mathtt{if}(\tfrac{1}{2}\mathtt{Coin}, x, \underline{0}))\right]\!\right]_0 = 1$$

$$\left[\!\left[\mathtt{fix}(\lambda x^{\mathtt{Int}}.\mathtt{if}((\mathtt{Rand})\,\underline{3}, x, \underline{0}))\right]\!\right]_0 = \tfrac{1}{2}.$$

FA relates Semantical and Observational equivalences:

Let $P, Q : A$

$$\forall a \in |A|,\ [\![P]\!]_a = [\![Q]\!]_a$$

Abstraction $\Downarrow$ $\Uparrow$ Fullness

$$\forall C : A \Rightarrow \mathtt{Int},\ \forall n \in |\mathtt{Int}|,\ \mathbf{Proba}((C)\,P \xrightarrow{*} n) = \mathbf{Proba}((C)\,Q \xrightarrow{*} n))$$

Abstraction proof:

1. Apply Adequacy Theorem : $\forall n,\ \mathbf{Proba}((C)\,P \xrightarrow{*} \underline{n}) = [\![(C)\,P]\!]_n$.
2. Apply Composition Formula:

$$\forall n,\ [\![(C)\,P]\!]_n = \sum_{m \in \mathcal{M}_f(|A|)} [\![C]\!]_{m,n} \prod_{a \in m} [\![P]\!]_a^{m(a)}$$

$$= \sum_{m \in \mathcal{M}_f(|A|)} [\![C]\!]_{m,n} \prod_{a \in m} [\![Q]\!]_a^{m(a)} = [\![(C)\,Q]\!]_n$$

Example: $\mathtt{fix}(\lambda x^{\mathtt{Int}}.\mathtt{if}(\frac{1}{2}\mathtt{Coin}, x, \underline{0})) \simeq_o 0$.

$$\left[\!\left[\mathtt{fix}(\lambda x^{\mathtt{Int}}.\mathtt{if}(\tfrac{1}{2}\mathtt{Coin}, x, \underline{0}))\right]\!\right]_0 = [\![\underline{0}]\!]_0 = 1.$$

FA relates Semantical and Observational equivalences:

Let $P, Q : A$

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Fullness proof:

1. By **contradiction**: $\exists a \in |A|\ ,\ [\![P]\!]_a \neq [\![Q]\!]_a$
2. Find **testing context**: $T_a : A \Rightarrow \texttt{Int}$ such that $[\![(T_a)\,P]\!]_0 \neq [\![(T_a)\,Q]\!]_0$
3. Prove **definability**: $T_a \in PPCF$
4. Apply **Adequacy**: $\textbf{Proba}((T_a)\,P \xrightarrow{*} 0) \neq \textbf{Proba}((T_a)\,Q \xrightarrow{*} 0)$.

Find a testing context: Base Case

Assumptions

- $P, Q : \texttt{Int}$
- $[\![P]\!]_n \neq [\![Q]\!]_n$

Goal

- $T_n : \texttt{Int} \Rightarrow \texttt{Int}$
- $[\![(T_n)\,P]\!]_0 \neq [\![(T_n)\,Q]\!]_0$

Choose

If $T_n = \lambda x^{\texttt{Int}}.\texttt{if}(x = \underline{n}, \underline{0}, \Omega_{\texttt{Int}})$ Then $\begin{cases} [\![T_n]\!]_{[n],0} = 1 \\ [\![T_n]\!]_{m,0} = 0 & \text{otherwise} \end{cases}$

Conclude by Composition Formula, $$[\![(T_n)\,P]\!]_0 = \sum_{m \in \mathcal{M}_f(|A|)} [\![T_n]\!]_{m,0} \prod_{a \in m} [\![P]\!]_a^{m(a)}$$

$$[\![(T_n)\,P]\!]_0 = [\![T_n]\!]_{[n],0} \prod_{a \in [n]} [\![P]\!]_a^{m(a)} = [\![P]\!]_n \neq [\![Q]\!]_n = [\![(T_n)\,Q]\!]_0$$

Find a testing context: Induction Case

Assumptions

- $P, Q : \mathtt{B} \Rightarrow \mathtt{C}$
- $a = ([b_1, \ldots, b_n], c)$
- $[\![P]\!]_a \neq [\![Q]\!]_a$

Goal:

- $T_a : (B \Rightarrow C) \Rightarrow \mathtt{Int}$
- $[\![(T_a)\, P]\!]_0 \neq [\![(T_a)\, Q]\!]_0$
- $[\![T_a]\!]_{(m,0)} \neq 0 \Leftrightarrow m = [a]$

Composition Formula:

$$[\![(T_a)\, P]\!]_0 = \sum_{m \in \mathcal{M}_f(|A|)} [\![T_a]\!]_{(m,0)} \prod_{a \in m} [\![P]\!]_a^{m(a)}$$

Choose:

$$\mathcal{P}(([b_1, \ldots, b_n], c))(\vec{X}) \doteq \lambda z^{B \Rightarrow C}.(\mathcal{P}(c))\, ((z)\, \mathtt{choose}(X_i \cdot \mathcal{N}(b_i))_{i=1}^n),$$

$$\mathcal{N}(([b_1, \ldots, b_n], c))(\vec{X}) \doteq \lambda x^B.\mathtt{if}(\wedge_{i=1}^n\, (\mathcal{P}(b_i))\, x, \mathcal{N}(c), \Omega_C).$$

Power Series:

$\forall m$, $\left[\!\!\left[\mathcal{P}(a)(\vec{X})\right]\!\!\right]_{(m,0)}$ is a power series in $\vec{X}$ with coeff of $\prod \vec{X} \neq 0 \iff m = [a]$

$\left[\!\!\left[\left(\mathcal{P}(a)(\vec{X})\right)\, P\right]\!\!\right]_0$ is a power series in $\vec{X}$ with

$\left[\!\!\left[\mathcal{P}(a)(\vec{X})\right]\!\!\right]_{([a],0)} [\![P]\!]_a$ as coeff of $\prod \vec{X}$.

Find a testing context: Induction Case

Assumptions

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$$\left[\!\left[\mathcal{P}(a)(\vec{X})\right]\!\right]_{([a],0)} [\![P]\!]_a \in \mathbb{R}^+ \cup \{\infty\}.$$

- Since $[\![P]\!]_a \neq [\![Q]\!]_a$, we have $\left[\!\left[\mathcal{P}(a)(\vec{X})\right]\!\right]_{([a],0)} [\![P]\!]_a \neq \left[\!\left[\mathcal{P}(a)(\vec{X})\right]\!\right]_{([a],0)} [\![Q]\!]_a$.
- $\left[\!\left[\left(\mathcal{P}(a)(\vec{X})\right) P\right]\!\right]$ and $\left[\!\left[\left(\mathcal{P}(a)(\vec{X})\right) Q\right]\!\right]$ are power series with distinct coefficients.

Definability: Find $\vec{\lambda} \in [0,1]^{(\mathbb{N})}$ then $\mathcal{P}(a)(\vec{\lambda})$ in PPCF such that $\left[\!\left[\left(\mathcal{P}(a)(\vec{\lambda})\right) P\right]\!\right]_0 \neq \left[\!\left[\left(\mathcal{P}(a)(\vec{\lambda})\right) Q\right]\!\right]_0$

By contradiction:

- If they were equal, their derivatives near zero would be equal.
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$$\left[\!\left[\mathcal{P}(a)(\vec{X})\right]\!\right]_{([a],0)} [\![P]\!]_a \in \mathbb{R}^+ \cancel{\cup \{\infty\}}. \Rightarrow \mathit{PCOH}$$

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Non degenerate Quantitative Semantics:

- **Probabilistic Coherence Spaces (Pcoh)**

J.-Y. Girard. Between logic and quantic: a tract. *Linear Logic in Comput. Sci., 2004.*

V. Danos, and T. Ehrhard. Probabilistic coherence spaces as a model of higher-order probabilistic computation. *Inf. Comput. Sci., 2011.*

- **Finiteness Spaces**

T. Ehrhard. Finiteness Spaces. *Math. Struct. Comput. Sci., 2005.*

- **Convenient Vector Spaces**

H. Kriegl, and P. Michor. The convenient setting of global analysis. *American Mathematical Society, 1997.*

R. Blute, T. Ehrhard, and C. Tasson. A convenient differential category. *Cahiers de topologies et de géométrie différentielles, 2012.*

M. Kerjean. Complete vector spaces as a quantitative and topological model of DiLL. *Master report, 2013.*

A Probabilistic Orthogonality

Orthogonality: $x, y \in (\mathbb{R}^+)^{|A|}$.

$$x \perp y \iff \sum_{a \in |A|} x_a y_a \in [0, 1].$$

Given a set $\mathrm{P} \subseteq (\mathbb{R}^+)^{|A|}$ we define $\mathrm{P}^\perp$, the *orthogonal* of P, as

$$\mathrm{P}^\perp \acute{=} \{y \in (\mathbb{R}^+)^{|A|} \mid \forall x \in \mathrm{P} \ \langle x \mid y \rangle \leq 1\}.$$

Probabilistic Coherence Space: $\mathcal{X} = (|\mathcal{X}|, \mathrm{P}(\mathcal{X}))$
where $|\mathcal{X}|$ is a countable set
and $\mathrm{P}(\mathcal{X}) \subseteq (\mathbb{R}^+)^{|\mathcal{X}|}$

such that the following holds:

closedness: $\mathrm{P}(\mathcal{X})^{\perp\perp} = \mathrm{P}(\mathcal{X})$,

boundedness: $\forall a \in |\mathcal{X}|, \exists \mu > 0, \forall x \in \mathrm{P}(\mathcal{X}), x_a \leq \mu$,

completeness: $\forall a \in |\mathcal{X}|, \exists \lambda > 0, \lambda e_a \in \mathrm{P}(\mathcal{X})$.

Types as Probabilistic Coherence Spaces

Objects of PCoh:

$\mathcal{X} = (|\mathcal{X}|, \mathrm{P}(\mathcal{X}))$
where $|\mathcal{X}|$ is a countable set
and $\mathrm{P}(\mathcal{X}) \subseteq (\mathbb{R}^+)^{|\mathcal{X}|}$

Type Example:

$$[\![\mathtt{Int}]\!] = \left(\mathbb{N}, \mathrm{P}(\mathtt{Int}) = \{(\lambda_n) \mid \sum_n \lambda_n \leq 1\}\right)$$

Data Example:

if $\mathtt{M} : \mathtt{nat}$, then $[\![\mathtt{M}]\!] \in \mathrm{P}(\mathtt{Int}) \subseteq (\mathbb{R}^+)^{\mathbb{N}}$
is a subprobability distributions.

Coin:

$\frac{1}{2} \cdot \underline{0} + \frac{1}{2} \cdot \underline{1}$

$$[\![\mathtt{Coin}]\!] = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0, \dots\right)$$

Programs as Probabilistic Coherence Maps

Maps of PCoh: $f : (|\mathcal{X}|, \mathrm{P}(\mathcal{X})) \to (|\mathcal{Y}|, \mathrm{P}(\mathcal{Y}))$

defined as a **matrix** $\mathrm{Mat}(f) \in (\mathbb{R}^+)^{\mathcal{M}_f(|\mathcal{X}|) \times |\mathcal{Y}|}$

thanks to **Composition Formula:**

$$f(x) = \sum_{m \in \mathcal{M}_f(|A|)} \mathrm{Mat}(f)_m \cdot x^m$$

with $x^m = \prod_{a \in \mathrm{Supp}(x)} x_a^{m(a)}$

f can be seen as an **entire function** $f : (\mathbb{R}^+)^{|\mathcal{X}|} \to (\mathbb{R}^+)^{|\mathcal{Y}|}$

preserving probabilistic coherence, $f(\mathrm{P}(\mathcal{X})) \subseteq \mathrm{P}(\mathcal{Y})$

Example: if $\mathtt{P} : \mathtt{Int} \Rightarrow \mathtt{Int}$, then $[\![\mathtt{P}]\!] : (\mathbb{R}^+)^{\mathbb{N}} \to (\mathbb{R}^+)^{\mathbb{N}}$ is an entire function preserving subprobability distributions.

A model of **Probabilistic PCF**,
same interpretation of terms as in the $\mathbb{R}^+$-weighted relational model.

Example of Programs in PCoh

$$[\![\texttt{Int}]\!] = \left(\mathbb{N}, \mathbf{P}\left(\texttt{Int}\right) = \{(\lambda_n) \mid \sum_n \lambda_n \leq 1\}\right)$$

`Once : Int ⇒ Int` λ`n if n=0 then 42 else Coin`

$$[\![\texttt{Once}]\!](x)_0 = \tfrac{1}{2}\sum_{n\geq 1} x_n$$
$$[\![\texttt{Once}]\!](x)_1 = \tfrac{1}{2}\sum_{n\geq 1} x_n$$
$$[\![\texttt{Once}]\!](x)_{42} = x_0$$

`Twice : Int ⇒ Int` λ`n if n=0 then 42 else Random n`

$$[\![\texttt{Twice}]\!](x)_k = \sum_{p=1}^{k}\sum_{q\geq k+1}\tfrac{1}{q}x_p x_q + \sum_{p=1}^{k+1}\sum_{q\geq k+1}(\tfrac{1}{p}+\tfrac{1}{q})x_p x_q, \text{ if } k \neq 42$$
$$[\![\texttt{Twice}]\!](x)_{42} = x_0 + \sum_{p=1}^{42}\sum_{q\geq 43}\tfrac{1}{q}x_p x_q + \sum_{p=1}^{43}\sum_{q\geq 43}(\tfrac{1}{p}+\tfrac{1}{q})x_p x_q$$

Non deterministic Orthogonality: $x, y \in \mathbb{K}^{|A|}$.

$$x \perp y \iff \sum_{a \in |A|} x_a y_a \leq 1 \text{ finite}$$
$$\iff |x| \cap |y| \text{ finite}$$

where $|x| = \{a \in |\mathcal{X}| \text{ s.t. } x_a \neq 0\}$.

Given a set $\mathrm{F} \in \mathcal{P}(|A|)$ we define $\mathrm{F}^{\perp}$, the *orthogonal* of F, as

$$\mathrm{F}^{\perp} \doteq \{u \subseteq |A| \mid \forall u \in \mathrm{F}\ u \cap v \text{ finite}\}.$$

$$\mathbb{K}\langle\mathcal{X}\rangle = \{v \in \mathbb{K}^{|\mathcal{X}|} \text{ s.t. } |v| \in \mathcal{F}(\mathcal{X})\} \subseteq \mathbb{K}^{|\mathcal{X}|}$$

Finiteness Space:

$\mathcal{X} = (|\mathcal{X}|, \mathrm{P}(\mathcal{X}))$,
with $\mathrm{P}(\mathcal{X}) \subseteq \overline{\mathbb{R}^+}^{|\mathcal{X}|}$
$\mathcal{X} = (|\mathcal{X}|, \mathcal{F}(\mathcal{X}))$
where $|\mathcal{X}|$ is a countable set
and $\mathcal{F}(\mathcal{X}) \subseteq \mathcal{P}(|\mathcal{X}|)$
such that $\mathcal{F}(\mathcal{X})^{\perp\perp} = \mathcal{F}(\mathcal{X})$,

Finitary Maps: $f : (|\mathcal{X}|, \mathcal{F}(\mathcal{X})) \to (|\mathcal{Y}|, \mathcal{F}(\mathcal{Y}))$

defined as a **matrix** $\mathrm{Mat}(f) \in \mathbb{K}^{\mathcal{M}_f(|\mathcal{X}|) \times |\mathcal{Y}|}$

thanks to **Composition Formula:**

$$f(x) = \sum_{m \in \mathcal{M}_f(|A|)} \mathrm{Mat}(f)_m \cdot x^m$$

with $x^m = \prod_{a \in \mathrm{Supp}(x)} x_a^{m(a)}$

f can be seen as an **entire function** $f : \mathbb{K}^{|\mathcal{X}|} \to \mathbb{K}^{|\mathcal{Y}|}$

~~**preserving** probabilistic coherence, $f(\mathrm{P}(\mathcal{X})) \subseteq \mathrm{P}(\mathcal{Y})$~~

preserving finitary coherence, $f(\mathbb{K}\langle\mathcal{X}\rangle) \subseteq \mathbb{K}\langle\mathcal{Y}\rangle$

$\mathbb{K}\langle\mathcal{X}\rangle$ is a **Topological Vector Space** with linearized topology.

A model of **Controlled Non determinism** with iteration but **no fixpoint**.

No basis:

- Quantitative semantics inspired by Linear algebra and Calculus.
- Interpret more complex data types than `bool` and `Int`.
- **Difficulty**: a cartesian closed category of vector spaces with infinite dimension.

Models of Differential and Classical Linear Logic

Objects : Complete Locally Convex Vector Spaces over $\mathbb{R}$ or $\mathbb{C}$.

Maps : Smooth maps, meaning preserving smooth curves.

or Holomorphic maps, meaning preserving holomorphic curves

or Power series, meaning converging sums of monomials

Conclusion

- $\mathfrak{R}$ **weighted relational models :**
  - Quantitative properties of programs are encoded by coefficients in $\mathfrak{R}$.
- PCOH a **fully abstract** model of probabilistic PCF:
  - Programms as power series
  - Semantics computed via intersection type system.
- **Topological models of classical and differential linear logic**:
  - Finiteness spaces (linearized topology)
  - Convenient vector spaces (usual topology, and new morphisms).