Directed Algebraic Topology and Concurrency
MSC 2014

A **Geometrical Interpretation** of **Asynchronous Computability**

joint ongoing work with
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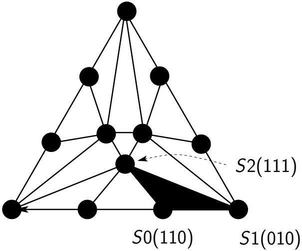
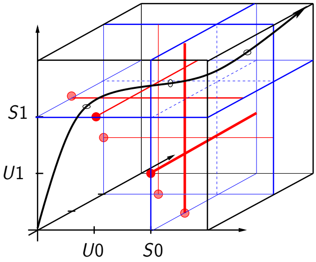
A Geometrical Interpretation of Asynchronous Computability

$U_1\ U_0\ S_1\ S_0\ U_2\ S_2$

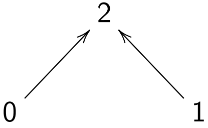
Interleaving Trace



Simplex



Dipath



Interval Order

Distributed System:

A fix family of $n+1$ processes communicate by **Update** and **Scan** of their **local** memory into a shared **global** memory.

Asynchronous:

- For each process, the kth Scan follows the kth Update
- Update and Scan are **mutually exclusive**
- no delay or order restriction

Interleaving Trace:

Each execution of a protocol is given by an **interleaving trace** $T \in \{U_i, S_i \mid i \in [n] = \{0 \cdots n\}\}^*$ well-bracketted.

3 processes, 2 rounds: $U_1\, U_2\, S_1\, U_0\, S_0\, S_2\, U_1\, U_0\, S_1\, U_2\, S_2\, S_0$

Consider a program with $n+1$ processes and $(r_i)_{i\in[n]}$ rounds.

State: a pair $s=(\ell,m)$ where

- $\ell=(\ell_i)_{i\in[n]}$ **local** memories (one register by process)
- $m=(m_i)_{i\in[n]}$ **global** memory (one register by process)

Initial state s_0: $\ell_i = i \quad \text{and} \quad m_i = \bot$

Semantics:

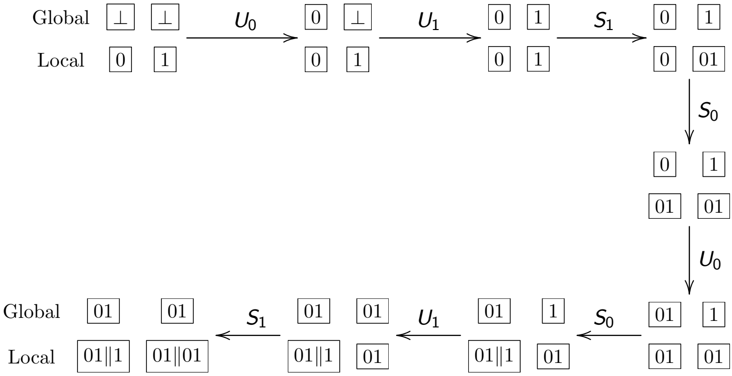
Update: i updates its local view into the global memory

$$(\ell_0 \ldots \ell_i \ldots \ell_n \,,\; m_0 \ldots \mathbf{m_i} \ldots m_n) \xrightarrow{U_i} (\ell_0 \ldots \ell_i \ldots \ell_n \,,\; m_0 \ldots \ell_{\mathbf{i}} \ldots m_n)$$

Scan: i scans the global memory into its local view

$$(\ell_0 \ldots \ell_{\mathbf{i}} \ldots \ell_n \,,\; m) \xrightarrow{S_i} (\ell_0 \ldots \mathbf{m} \ldots \ell_n \,,\; m)$$

2 processes, 2 rounds: $U_0\ U_1\ S_1\ S_0\ U_0\ S_0\ U_1\ S_1$



Definition:

Two interleaving traces T, T' are operationaly equivalent when

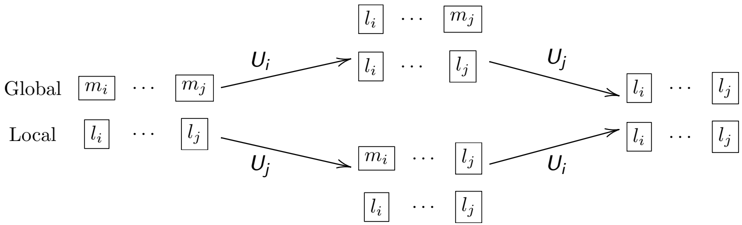
$$s_0 \xrightarrow{T}^{*} s \qquad \text{iff} \qquad s_0 \xrightarrow{T}^{*} s$$

Generators:

The interleaving trace equivalence $\approx$ is the smallest congruence on well-bracketed words in $\{U_i, S_i \mid i \in [n]\}^*$ such that

$$U_i U_j \approx U_j U_i \qquad \text{and} \qquad S_i S_j \approx S_j S_i$$

Proof Sketch:



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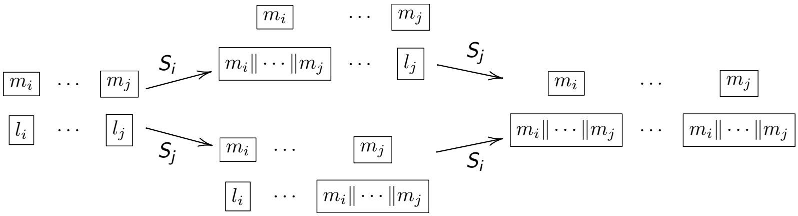
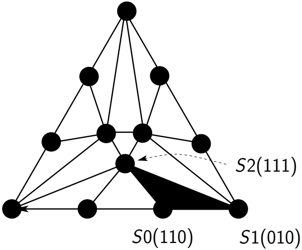


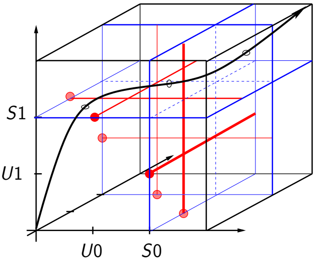
A Geometrical Interpretation of Asynchronous Computability

$[U_1\ U_0\ S_1\ S_0\ U_2\ S_2]$

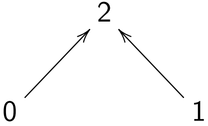
Interleaving Trace$/\approx$



Simplex

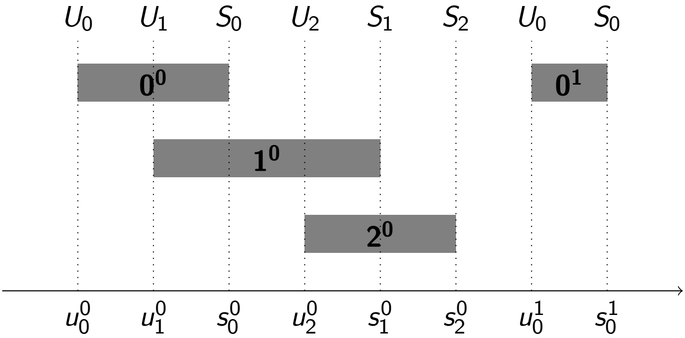


Dipath$/\leftrightsquigarrow$

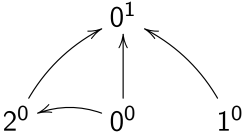


Interval Order

From Interleaving Traces to Interval Order



$$i^k \prec j^\ell \quad \text{iff} \quad s_i^k < u_j^\ell$$

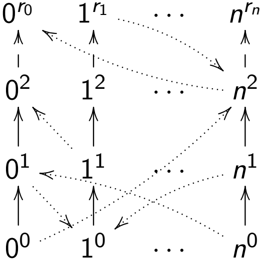


Interval Order

Consider a program with $n+1$ processes and $(r_i)_{i\in[n]}$ rounds.

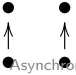
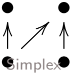
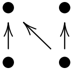
[n]-Colored Interval Order: $X = \{ i^k \mid k \in [r_i],\ i \in [n] \}$ with

- a partial order $\prec$ induced by **intervals** $i^k = [u_i^k, s_i^k]$
- restriction to any process i is a **total** order:

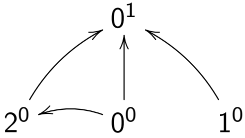


$$
\begin{cases}
i^k \prec j^l \quad \text{iff} \quad s_i^k < u_j^l \\
u_i^k < s_i^k \\
i^k \prec i^{k+1}
\end{cases}
$$

Theorem [Fishburn]: Interval orders are exactly the $(2+2)$-free posets,

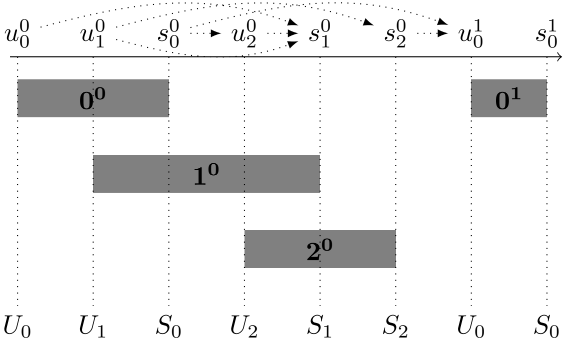
 implies  or 

From Interval Order to Interleaving Traces



$$\boxed{i^k \prec j^\ell \quad \text{iff} \quad s_i^k < u_j^\ell}$$

$$\begin{aligned} i^k \prec j^\ell &\Rightarrow s_i^k < u_j^\ell \\ i^k \| j^\ell &\Rightarrow s_i^k > u_j^\ell \quad \text{and} \quad s_j^\ell > u_i^k \end{aligned}$$



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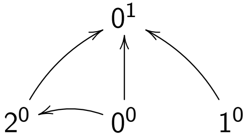

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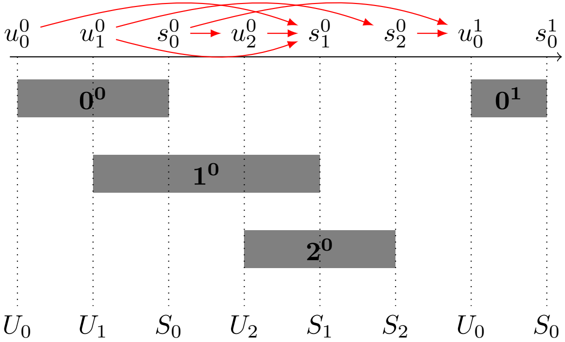

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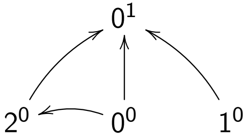

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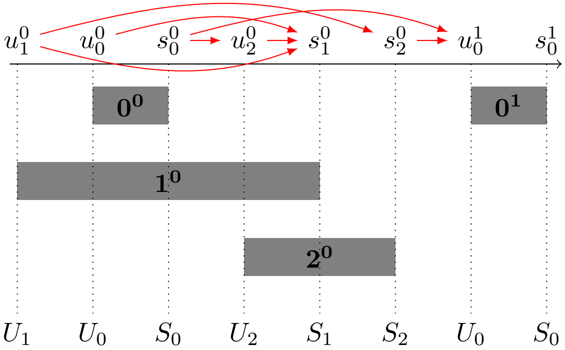

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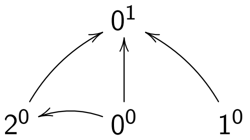

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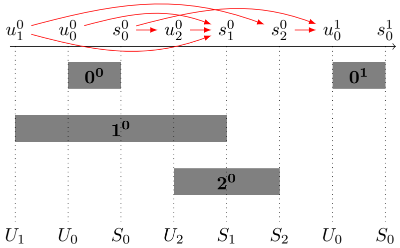

Remark: Relative position of Ss and Us are fixed.

Proposition: Interval Order induces equivalent interleaving traces.

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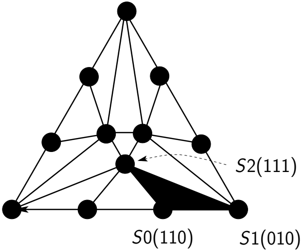
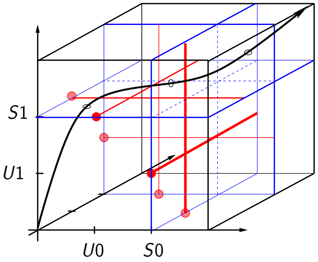
Interleaving Trace$/\approx$



Simplex



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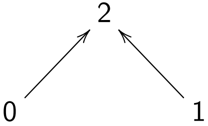
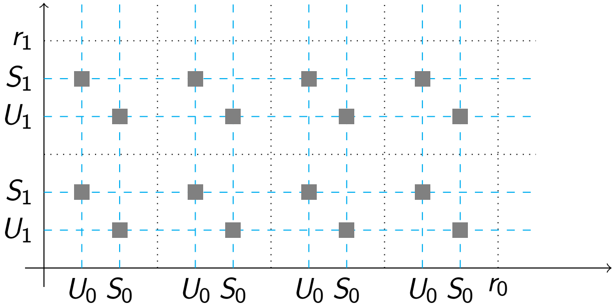


Interval Order

Pospace: $\mathbb{X}_n = \prod_{i \in [n]} [0, r_i] \setminus \bigcup_{\substack{i,j \in [n] \\ k \in [r_i],\ l \in [r_j]}} U_i^k \cap S_j^l$



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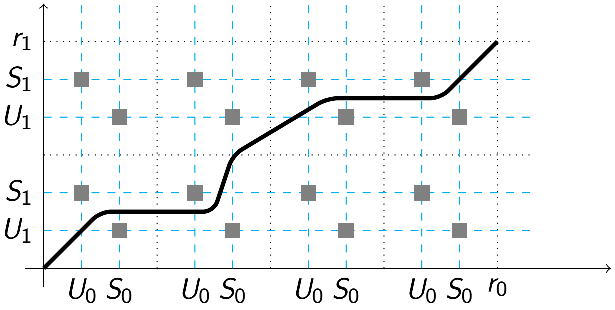
Dipath: $\alpha : [0,1] \to \mathbb{X}_n$ continuous and non decreasing



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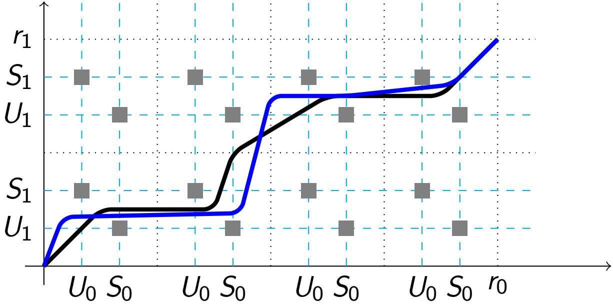
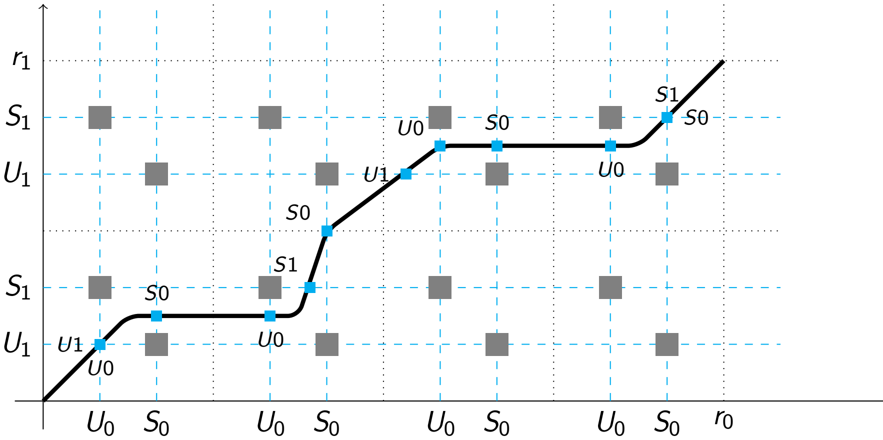
Dipath: $\alpha : [0,1] \to \mathbb{X}_n$ continuous and non decreasing

Dihomotopy: $h : \overrightarrow{[0,1]} \times [0,1] \to \mathbb{X}_n$ continuous non decreasing



Intersection with Update and Scan hyperplanes:



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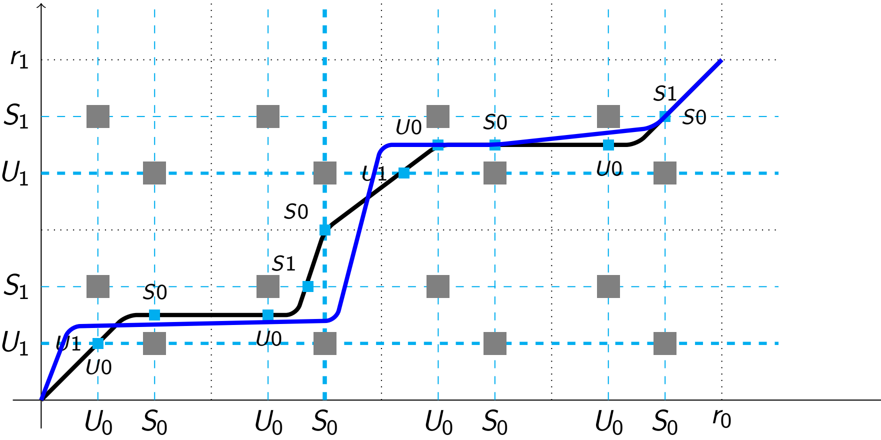

Interval Order: Characterized by relative position of U and S,

$$U_1^0 < S_0^1 < U_1^1$$

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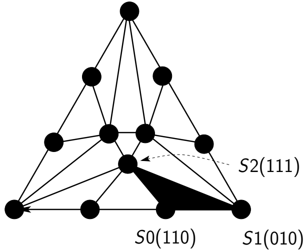
Interleaving Trace$/\approx$



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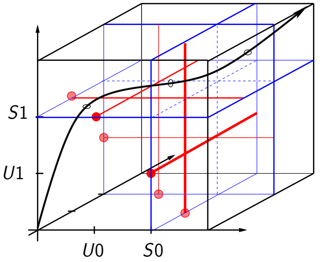

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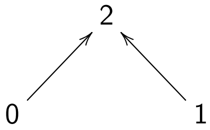


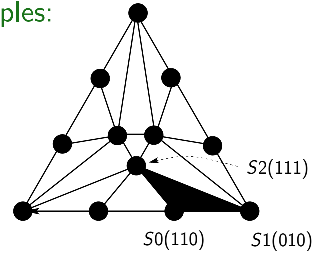
Interval Order

Consider a program with $n+1$ processes and $(r_i)_{i\in[n]}$ rounds.

Complex of executions:

- **Vertex:** (process, local memory)
- **Maximal Simplex:** $\{(0,\ell_0),\ldots,(n,\ell_n)\}$ where ℓ_i is the local view by process i of the global execution.

Examples:



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Examples:



$$0,0\bot \xrightarrow[0\,\rightarrowtail\,1]{} 1,01 \xrightarrow[0\quad 1]{} 0,01 \xrightarrow[1\,\rightarrowtail\,0]{} 1,\bot 1$$

Global	$\bot$	$\bot$	$\xrightarrow{U_0}$	0	$\bot$	$\xrightarrow{S_0}$	0	$\bot$	$\xrightarrow{U_1}$	0	1	$\xrightarrow{S_1}$	0	1
Local	0	1		0	1		$0\bot$	1		$0\bot$	1		$0\bot$	01

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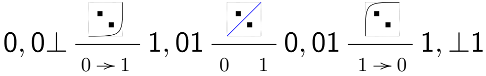


			$\xrightarrow{U_0}$			$\xrightarrow{U_1}$			$\xrightarrow{S_1}$			$\xrightarrow{S_0}$		
Global	⊥	⊥		0	⊥		0	1		0	1		0	1
Local	0	1		0	1		0	1		0	01		01	01

Operational Semantics: The ith local memory contains all the Updates preceding the last ith Scan.

Interval Order: $i^k \prec j^\ell \quad \text{iff} \quad S_i^j < U_k^\ell$

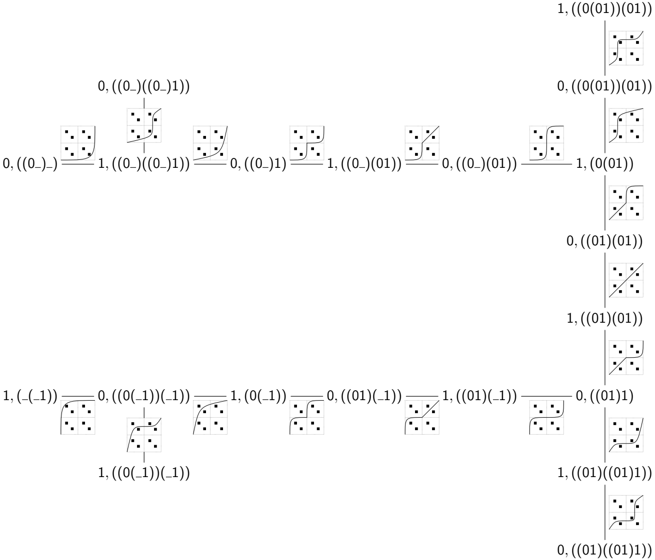
$$S_i^k > U_j^\ell \qquad \text{iff} \qquad i^k \parallel j^\ell \quad \text{or} \quad j^\ell \prec i^k \tag{1}$$

Asynchronous Complex: n processes, $(r_i)_{i\in[n]}$ rounds

- **Vertex:** (i^k, V_i^k) with V_i^k interval order satisfying (1),
- **Maximal Simplex:** $\{(0^{r_0}, V_0^{r_0}), \ldots, (n^{r_n}, V_n^{r_n})\}$
  if there is $X_n = \{j^\ell \mid j \in [n],\ \ell \in [r_i]\}$ an interval order its restriction to i^k is

$$V_i^k = \left\{ j^\ell \;\middle|\; i^k \parallel j^\ell \text{ or } j^\ell \prec i^k \right\}$$

2 **processes,** 2 **rounds:** (no layer, no immediate snapshot)



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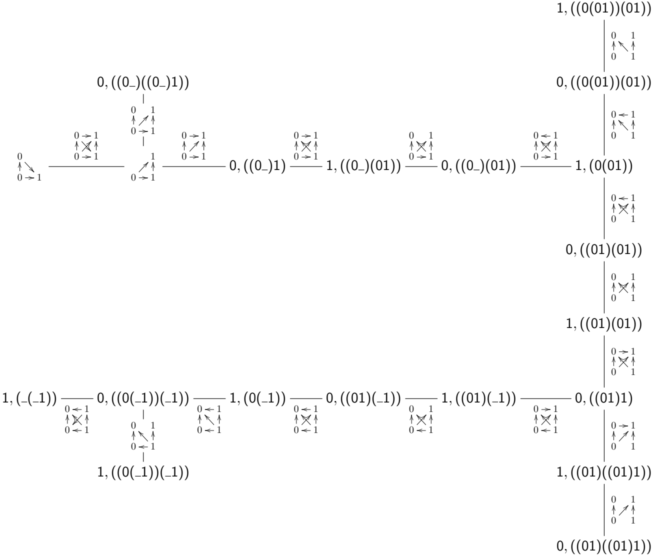
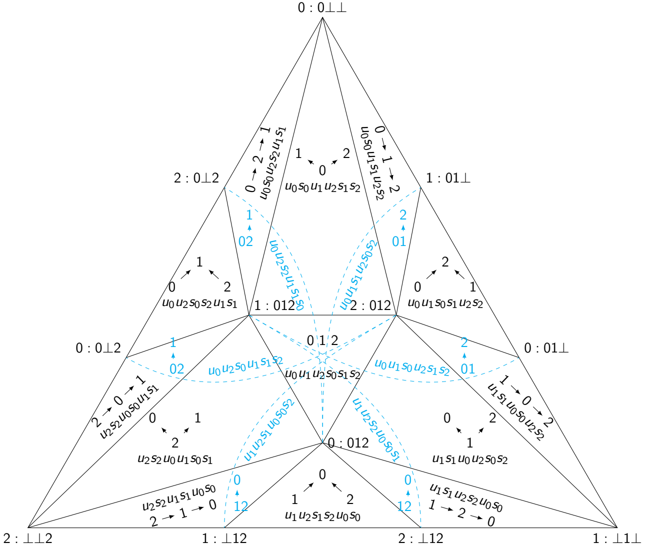


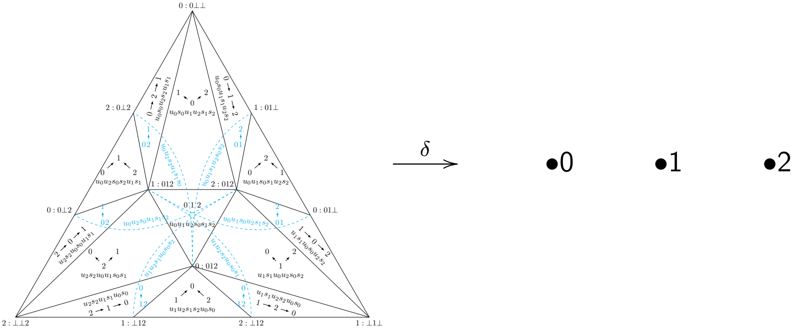
3 **processes,** 1 **rounds:** (no layer, no immediate snapshot)



Theorem [Herlihy & al.]: If the Protocol Complex is **contractible** then, the consensus is impossible.

Proof sketch:

Assume there is an algorithm δ solving the task, for any execution.

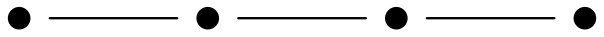


Theorem [Kozlov]:

Chromatic subdivision is collapsible, thus contractible.

Free Face: $\tau \subsetneq \sigma$ in K, with σ the **only** such maximal simplex.

Collapse: Take off a free face $\tau \subset \sigma$ and in between simplexes.
K is **collapsible** if there is a collapse sequence to the point.



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$\bullet$ —— $\bullet$ —— $\bullet$ —— $\bullet$ $\quad\Rightarrow\quad$ $\circ$ —— $\bullet$ —— $\bullet$ —— $\circ$

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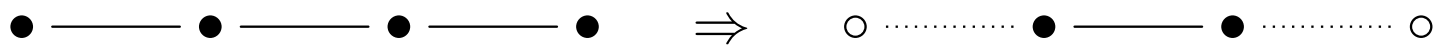

An other proof of Collapsibility

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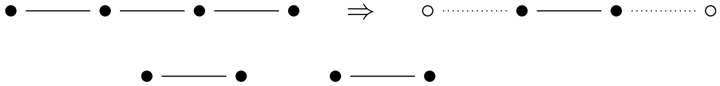

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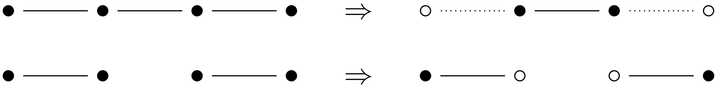

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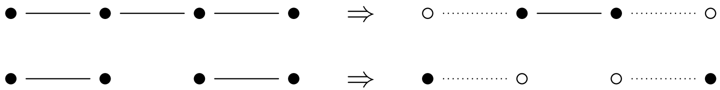

An other proof of Collapsibility

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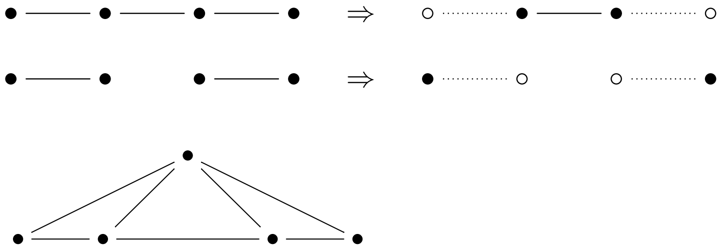

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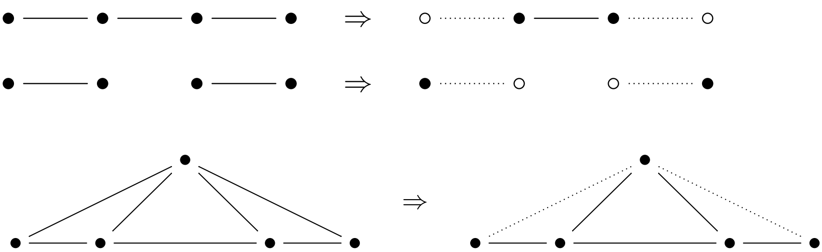

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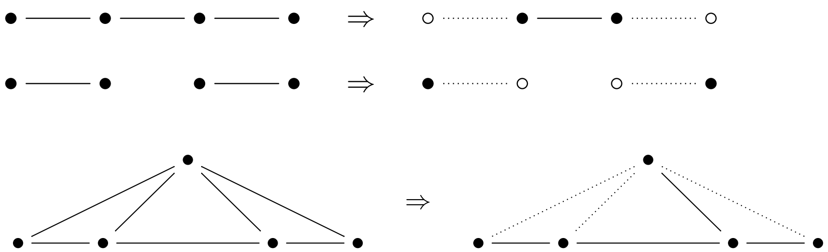

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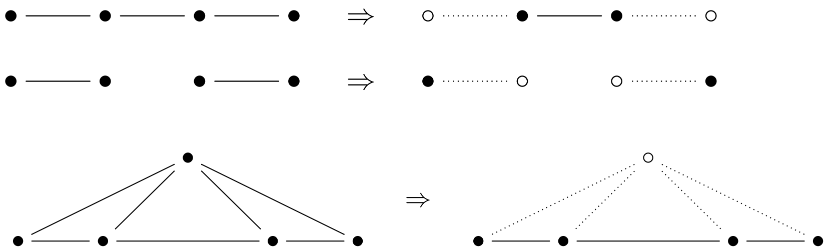

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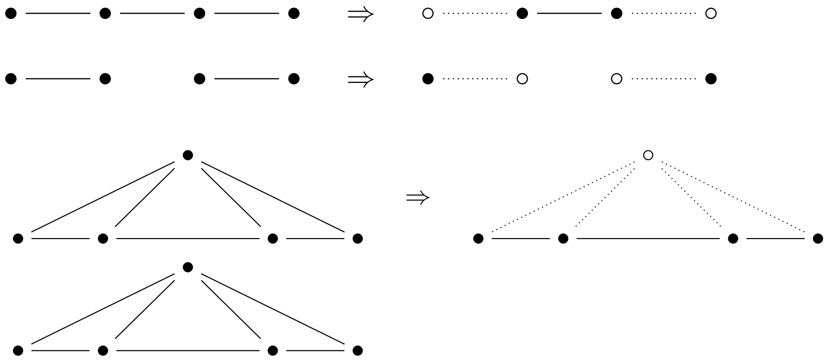

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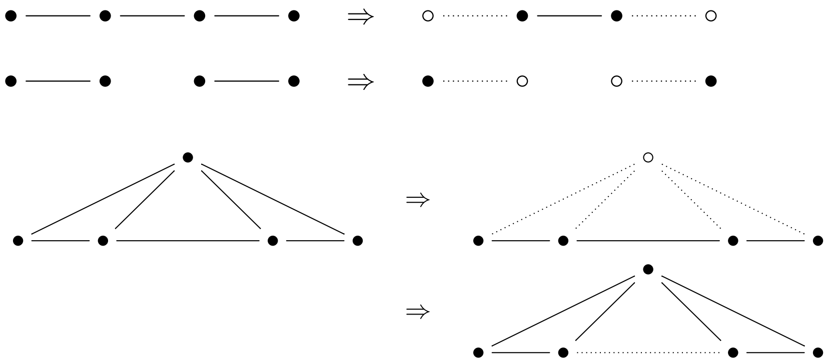

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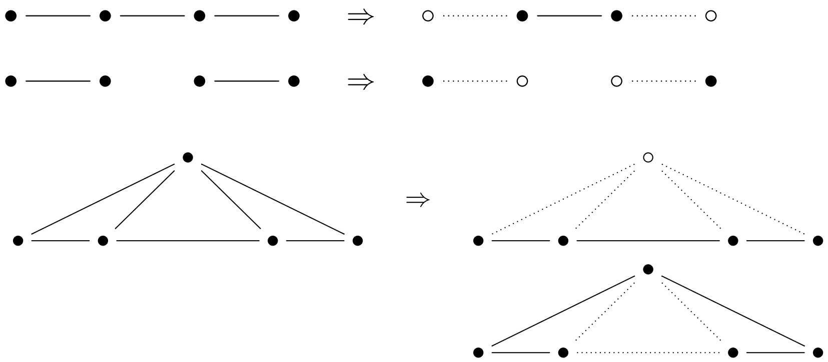

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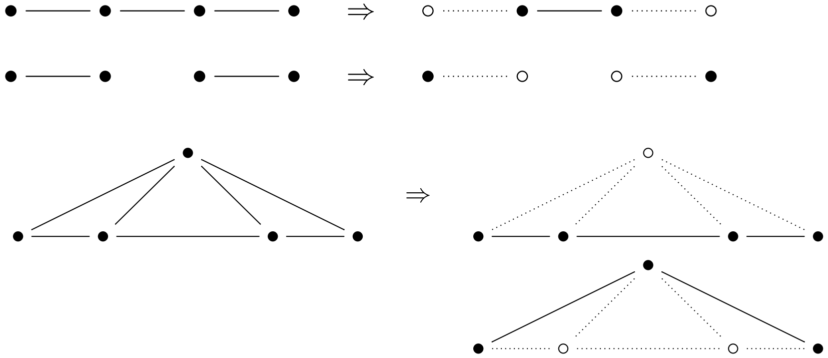
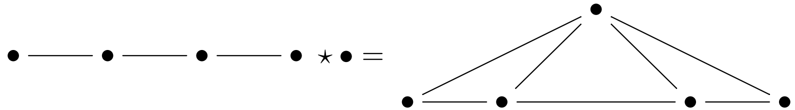
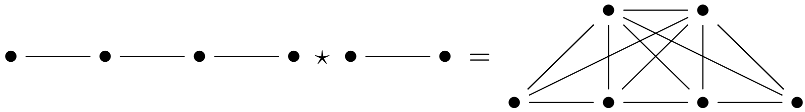


Collapses and (Colored) Joins

Join:



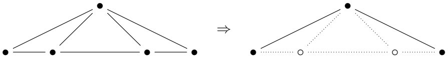


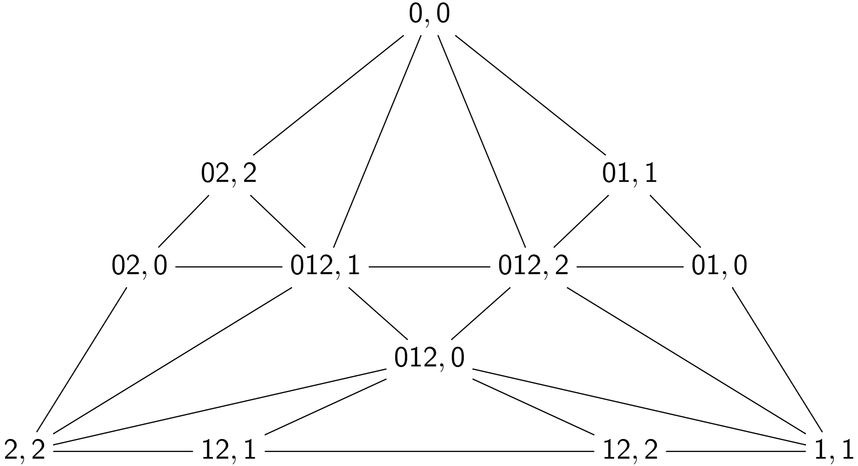
Collapses:

$$\chi(\Delta^I) \star \Delta^J \Rightarrow \partial\, \chi(\Delta^I) \star \Delta^J$$

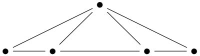
$$\chi(\Delta^I) \star \Delta^J \Rightarrow \chi(\Delta^I) \star \partial \Delta^J$$

Collapsibility of Iterated Protocol Complex

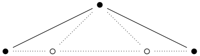


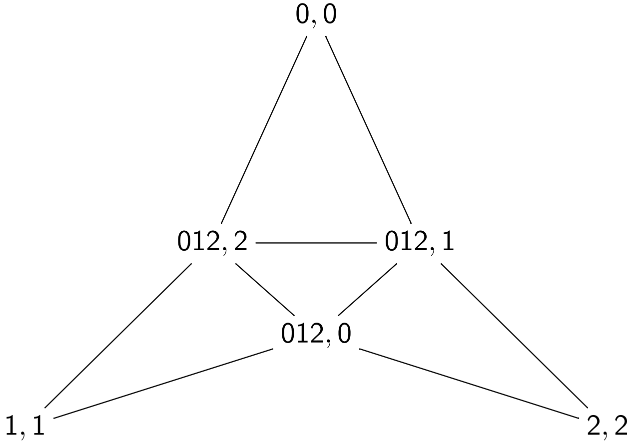


Collapsibility of Iterated Protocol Complex

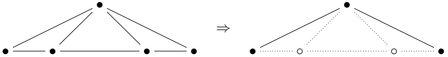


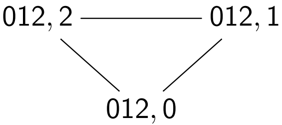
$\Rightarrow$



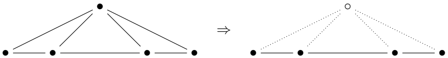


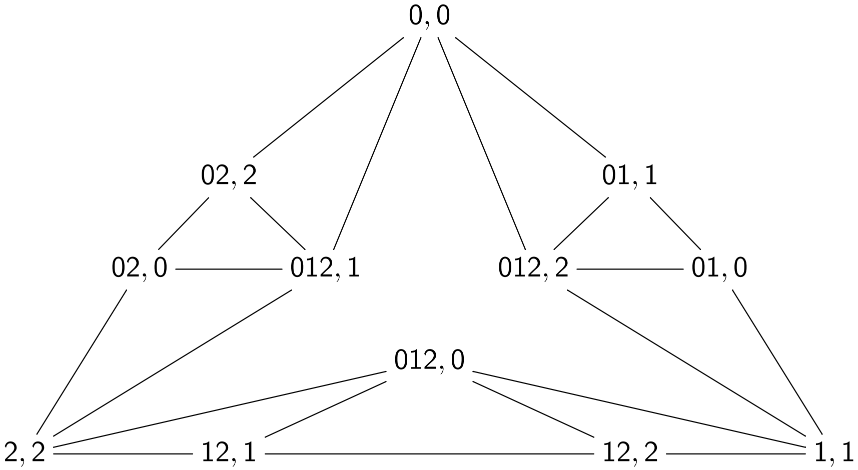
Collapsibility of Iterated Protocol Complex



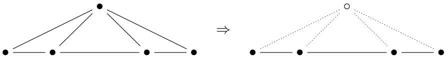


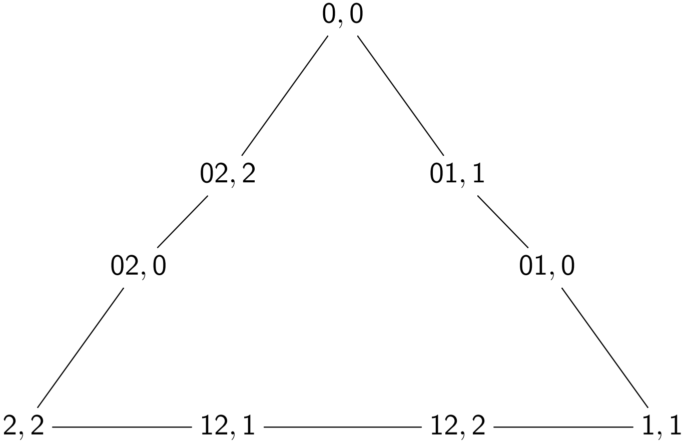
Collapsibility of Iterated Protocol Complex





Collapsibility of Iterated Protocol Complex

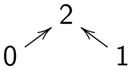




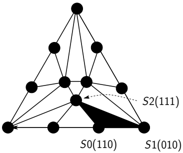
Equivalent presentations of Asynchronous Computations:

$U_1\ U_0\ S_1\ S_0\ U_2\ S_2$

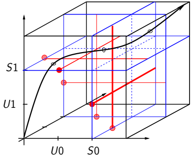
Interleaving Trace$/\approx$



Interval Order



Simplex



Dipath

Collapsing path of iterated protocol complex: a procedure

What's next:

- Such equivalence for other models of communication
- Compare collapsing procedure with Kozlov procedure
- Translate collapsing path into pospace (link with Trace Space [Raussen]).