The linear-non-linear substitution 2-monad

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Abstract

We introduce a general construction on 2-monads. We develop background on maps of 2-monads, their left semi-algebras, and colimits in 2-category. Then we introduce the construction of a colimit induced by a map of 2-monads, show that we obtain the structure of a 2-monad and give a characterisation of its algebras. Finally, we apply the construction to the map of 2-monads between free symmetric monoidal and the free cartesian 2-monads and combine them into a linear-non-linear 2-monad.

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This paper is concerned with a particular general construction on 2-monads in the sense of Cat-enriched monad theory [7]. Prima facie, the construction is not a universal one in a standard 2-category of 2-monads. All the same we are able precisely to characterise the 2-category of algebras for the 2-monad which we construct. This is a first step and further work will involve 2-dimensional monad theory in the sense of [4]. Specifically, we shall address the question of extending our constructed 2-monads on the 2-category **Cat** of small categories to the corresponding bicategory **Prof** of profunctors or distributeurs [2, 6, 1]. We shall then use a resulting Kleisli bicategory [12] as the setting for an analysis of the foundations of the differential calculus as it appears in the differential λ-calculus [8, 5, 10]. This will involve an extension of the approach of variable binding and substitution in abstract syntax [21, 9, 11, 15, 17].

Our project is based on 2-monads on a 2-category $\mathbf{K}$ in the setting of the pioneering paper [4]. Here, for a 2-monad $\mathcal{T}$ on $\mathbf{K}$, we follow the practice of that paper in writing $\mathcal{T}\text{-}\mathbf{Alg}_s$ for the 2-category of strict $\mathcal{T}$-algebras, strict $\mathcal{T}$-algebra maps and $\mathcal{T}$-algebra 2-cells. We shall use more detailed information from [4] in further papers.

In (enriched) categories of algebras for a monad, limits are easy and it is colimits which are generally of more interest. We assume throughout that our ambient 2-category $\mathbf{K}$ is cocomplete, that our 2-monads $\mathcal{T}$ are such that the 2-categories $\mathcal{T}\text{-}\mathbf{Alg}_s$ are also cocomplete. In fact, we shall only need rather innocent looking colimits in $\mathcal{T}\text{-}\mathbf{Alg}_s$, specifically the co-lax colimit of an arrow. However, even that requires an infinite construction [18]. So it does not seem worth worrying about minimal conditions for our results: we assume that we are in a situation where all our 2-categories are cocomplete. That happens for example if our basic 2-category is locally finitely presentable and our monads are finitary [19].

Content

We first describe the background in Section 1 on maps of 2-monads (Subsection 1.1), left-semi algebras (Subsection 1.2) and colimits (Subsection 1.3), needed in our main Section 2. We first define the colimits obtained from a map of monads (Subsection 2.1) and exhibit their properties (Subsection 2.2). Inspired by these properties, we define what we simply call the

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Structure 2-category (Subsection 2.3). We finally use (Subsection 2.4) the properties of the Structure 2-category to prove, in Theorem 22 that the colimit is a monad; and finally we prove our main Theorem 25 which states that the Structure 2-category is isomorphic to the 2-category of strict algebras over the colimit monad. We end by spelling out the construction for two examples, the first one generates the left-semi algebra 2-category (Proposition 26) and the second the linear-non-linear monad (Section 3) which was the original intention for developing this theory.

Notations

We denote as $[n]$ the set $\{1, \ldots, n\}$ for $n \in \mathbf{N}$. In a 2-category $\mathbf{K}$, we denote as 1_Z the identity 1-cell on the object Z and horizontal composition as $g\,f$ for $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$; we denote as id_f the identity 2-cell on the morphism f and the vertical composition as $\beta * \alpha$ for 2-cells $\alpha : g \Rightarrow g'$ and $\beta : g' \Rightarrow g''$. We denote as $\alpha.f$ the horizontal composition of α and id_f.

1 Background

1.1 Maps of 2-monads

The construction which we introduce here takes for its input a map $\lambda : \mathcal{L} \Rightarrow \mathcal{M}$ of 2-monads on $\mathcal{K}$. For clarity we stress that the usual diagrams commute on the nose. We rehearse some folklore related to this situation.

First, it is elementary categorical algebra that the monad map $\lambda : \mathcal{L} \Rightarrow \mathcal{M}$ induces a 2-functor $\lambda^* : \mathcal{L}\text{-}\mathbf{Alg}_s \Rightarrow \mathcal{M}\text{-}\mathbf{Alg}_s$ On objects λ^* takes an $\mathcal{M}$-algebra $\mathcal{M}X \to X$ to an $\mathcal{L}$-algebra $\mathcal{L}X \xrightarrow{\lambda} \mathcal{M}X \to X$. It is equally evident that $\lambda : \mathcal{L} \Rightarrow \mathcal{M}$ induces a 2-functor $\lambda_! : \mathbf{kl}(\mathcal{L}) \Rightarrow \mathbf{kl}(\mathcal{M})$ between the corresponding Kleisli 2-categories. We have the standard locally full and faithful comparisons: $\mathbf{kl}(\mathcal{L}) \to \mathcal{L}\text{-}\mathbf{Alg}_s$ and $\mathbf{kl}(\mathcal{M}) \to \mathcal{M}\text{-}\mathbf{Alg}_s$.

Suppose we interpret $\lambda_!$ as acting on the free algebras so that $\lambda_!$ takes the free $\mathcal{L}$-algebra $\mathcal{L}^2 A \xrightarrow{\mu^{\mathcal{L}}} \mathcal{L}A$ to the free $\mathcal{M}$-algebra $\mathcal{M}^2 A \xrightarrow{\mu^{\mathcal{M}}} \mathcal{M}A$. Then we can see $\lambda_!$ as a restricted left adjoint to λ^* in the following sense. Given the free $\mathcal{L}$-algebra $\mathcal{L}^2 \xrightarrow{\mu^{\mathcal{L}}} \mathcal{L}A$ on A and $\mathcal{M}B \xrightarrow{b} B$ an arbitrary $\mathcal{M}$-algebra, we have $\mathcal{L}\text{-}\mathbf{Alg}_s(\mathcal{L}A, \lambda^* B) \simeq \mathcal{M}\text{-}\mathbf{Alg}_s(\lambda_! \mathcal{L}A, B)$. For $\lambda_!(\mathcal{L}^2 A \xrightarrow{\mu^{\mathcal{L}}} \mathcal{L}A) = \mathcal{M}^2 A \xrightarrow{\mu^{\mathcal{M}}} \mathcal{M}A$ and so both sides are isomorphic to $\mathcal{K}(A, B)$.

Any $\mathcal{L}$-algebra $\mathcal{L}A \xrightarrow{a} A$ lies in a coequalizer diagram in $\mathcal{L}\text{-}\mathbf{Alg}_s$: $\mathcal{L}^2 A \underset{\mathcal{L}a}{\overset{\mu^{\mathcal{L}}}{\rightrightarrows}} \mathcal{L}A \xrightarrow{a} a$. So to extend $\lambda_!$ to a full left adjoint $\lambda_! : \mathcal{L}\text{-}\mathbf{Alg}_s \to \mathcal{M}\text{-}\mathbf{Alg}_s$ one has only to take the coequalizer of the corresponding pair in $\mathcal{M}\text{-}\mathbf{Alg}_s$: $\mathcal{M}\mathcal{L}A \underset{\mathcal{M}a}{\overset{\mu^{\mathcal{M}}\mathcal{M}\lambda}{\rightrightarrows}} \mathcal{M}A$. As it happens, we do not need the full left adjoint, but we shall need the unit of the adjunction given by the $\mathcal{L}$-algebra map λ_A from $\mathcal{L}^2 A \xrightarrow{\mu^{\mathcal{L}}} \mathcal{L}A$ to $\lambda^*\lambda_!(\mathcal{L}^2 A \xrightarrow{\mu^{\mathcal{L}}} \mathcal{L}A) = \mathcal{L}\mathcal{M}A \xrightarrow{\lambda\mathcal{M}} \mathcal{M}^2 A \xrightarrow{\mu^{\mathcal{M}}} \mathcal{M}A$.

If $\begin{tikzcd} \mathcal{L}A \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{L}B \end{tikzcd}$ is an $\mathcal{L}$-algebra 2-cell then the corresponding 2-cell $\lambda^*\lambda_! g \Rightarrow \lambda^*\lambda_! g$ is given by the composite $\begin{tikzcd} \mathcal{M}A \arrow[r, "\mathcal{M}\eta\mathcal{L}"] & \mathcal{M}\mathcal{L}A \arrow[r, bend left, "\mathcal{M}g"] \arrow[r, bend right, "\mathcal{M}g'"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{M}\mathcal{L}B \arrow[r, "\mathcal{M}\lambda"] & \mathcal{M}^2B \arrow[r, "\mu^{\mathcal{M}}"] & \mathcal{M}B \end{tikzcd}$ so that

$$\begin{tikzcd} \mathcal{L}A \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] \arrow[d, "\lambda"] & \mathcal{L}B \arrow[d, "\lambda"] \\ \mathcal{M}A \arrow[r, bend left, "\lambda^*\lambda_!g"] \arrow[r, bend right, "\lambda^*\lambda_!g"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{M}B \end{tikzcd} \quad \text{i.e.} \quad \begin{tikzcd} \mathcal{L}A \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{L}B \arrow[d, "\lambda"] \\ & \mathcal{M}A \end{tikzcd} = \begin{tikzcd} \mathcal{L}A \arrow[d, "\lambda"] & \\ \mathcal{M}A \arrow[r, bend left, "\lambda^*\lambda_!g"] \arrow[r, bend right, "\lambda^*\lambda_!g"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{M}B \end{tikzcd} \tag{1}$$

1.2 Left-semi Algebras

In this section we present a theory of a generalization of the notion of $\mathcal{T}$-algebra for a 2-monad $\mathcal{T}$. In effect, it is a mere glimpse of an extensive theory of semi-algebra structure, in the sense of structure "up to a retraction", a terminology well-established in computer science. We do not need to have this background in place for the results which we give in this paper: we give only what is required to make the paper comprehensible. However, some impression of what is involved can be obtained by looking at [14] which gives some theory in the 1-dimensional context.

▶ **Definition 1.** *Let $\mathcal{T}$ be a 2-monad on a 2-category $\mathbf{C}$. A left-semi $\mathcal{T}$-algebra structure on an object Z of $\mathbf{C}$ consists of a 1-cell $\mathcal{T}Z \xrightarrow{z} Z$ and a 2-cell $\epsilon : z.\eta \Rightarrow 1_Z$ satisfying the following 1-cell and 2-cell equalities:*

$$\begin{tikzcd} \mathcal{T}^2Z \arrow[r, "\mathcal{T}z"] \arrow[d, "\mu"'] & \mathcal{T}Z \arrow[d, "z"] \\ \mathcal{T}Z \arrow[r, "z"] & Z \end{tikzcd} \tag{2}$$

$$\begin{tikzcd} \mathcal{T}Z \arrow[d, "z"] & \\ Z \arrow[r, "\eta"] \arrow[dr, equal] & \mathcal{T}Z \arrow[d, "z"] \arrow[dl, phantom, "\Leftarrow\epsilon"] \\ & Z \end{tikzcd} = \begin{tikzcd} \mathcal{T}Z \arrow[d, bend right, "z"'] \arrow[d, bend left, "z"] \arrow[d, phantom, "="] \\ Z \end{tikzcd} = \begin{tikzcd} \mathcal{T}Z \arrow[r, "\mathcal{T}\eta"] \arrow[dr, equal] & \mathcal{T}^2Z \arrow[d, "\mathcal{T}z"] \arrow[dl, phantom, "\Leftarrow\mathcal{T}\epsilon"] \\ & \mathcal{T}Z \arrow[d, "z"] \\ & Z \end{tikzcd} \tag{3}$$

▶ Remark 2. **1.** The diagrams

$$\begin{tikzcd} \mathcal{T}Z \arrow[r, "\eta\mathcal{T}"'] \arrow[rr, bend left, equal] \arrow[d, "z"'] & \mathcal{T}^2Z \arrow[r, "\mu"'] \arrow[d, "\mathcal{T}z"] & \mathcal{T}Z \arrow[d, "z"] \\ Z \arrow[r, "\eta"] & \mathcal{T}Z \arrow[r, "z"] & Z \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{T}Z \arrow[r, "\eta\mathcal{T}"'] \arrow[rr, bend left, equal] & \mathcal{T}^2Z \arrow[r, "\mu"'] \arrow[d, "\mathcal{T}z"] & \mathcal{T}Z \arrow[d, "z"] \\ & \mathcal{T}Z \arrow[r, "z"] & Z \end{tikzcd}$$

demonstrate that Condition (2) implies that the boundaries of the 2-cells in (3) do match.

2. Condition (2) is the standard composition for a strict $\mathcal{T}$-algebra, while Condition (3) is the unit condition for a colax $\mathcal{T}$-algebra.

▶ **Definition 3.** *Suppose that $\mathcal{T}Z \xrightarrow{z} Z, \epsilon : z.\eta \Rightarrow 1_Z$ and $\mathcal{T}W \xrightarrow{w} W, \epsilon : w.\eta \Rightarrow 1_W$ are left-semi $\mathcal{T}$-algebras. A strict map from the first to the second consists of $p : Z \to W$ satisfying the following 1-cell and 2-cell equalities:*

$$\begin{tikzcd} \mathcal{T}Z \arrow[r, "\mathcal{T}p"] \arrow[d, "z"'] & \mathcal{T}W \arrow[d, "w"] \\ Z \arrow[r, "p"] & W \end{tikzcd} \tag{4}$$

$$\begin{tikzcd} Z \arrow[r, "p"] & W \arrow[r, "\eta"] \arrow[dr, equal] & \mathcal{T}W \arrow[d, "w"] \arrow[dl, phantom, "\Leftarrow\epsilon"] \\ & & W \end{tikzcd} = \begin{tikzcd} Z \arrow[r, "\eta"] \arrow[dr, equal] & \mathcal{T}Z \arrow[d, "z"] \arrow[dl, phantom, "\Leftarrow\epsilon"] & \\ & Z \arrow[r, "p"] & W \end{tikzcd} \tag{5}$$

▶ Remark 4. **1.** The Condition (4) with the naturality of η imply that the boundaries of the 2-cells in (5) do match.

2. The definition is the restriction to left-semi algebras of the evident notion of strict map of colax $\mathcal{T}$-algebras.
3. If $\mathcal{T}Z \xrightarrow{z} Z, \epsilon : z.\eta \Rightarrow 1_Z$ is a left-semi algebra, then $\mathcal{T}Z \xrightarrow{z} Z$ is a strict map to it from the free algebra $\mathcal{T}^2Z \xrightarrow{\mu} \mathcal{T}Z$.

▶ **Proposition 5.** *Suppose that $\mathcal{T}Z \xrightarrow{z} Z, \epsilon : z.\eta \Rightarrow 1_Z$ is a left-semi algebra. Then the composite $f : Z \xrightarrow{\eta} \mathcal{T}Z \xrightarrow{z} Z$ is a strict endomap of the left-semi algebra.*

Finally, we consider 2-cells between maps of left-semi algebras.

▶ **Definition 6.** *Suppose that $p, q : Z \to W$ are strict maps of left-semi algebras from $\mathcal{T}Z \xrightarrow{z} Z, \epsilon : z.\eta \Rightarrow 1_Z$ to $\mathcal{T}W \xrightarrow{w} W, \epsilon : w.\eta \Rightarrow 1_W$. A 2-cell from p to q consists of a 2-cell $\gamma : p \Rightarrow q$ such that the equality* $\begin{tikzcd} \mathcal{T}Z \arrow[r, "z"] & Z \arrow[r, bend left, "p"] \arrow[r, bend right, "q"'] \arrow[r, phantom, "\Downarrow\gamma"] & W \end{tikzcd} = \begin{tikzcd} \mathcal{T}Z \arrow[r, bend left, "\mathcal{T}p"] \arrow[r, bend right, "\mathcal{T}q"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathcal{T}W \arrow[r, "w"] & W \end{tikzcd}$ *holds.*

▶ Remark 7. Again, this is simply the restriction to the world of left-semi algebras of the definition of 2-cells for colax algebras.

▶ **Proposition 8.** *Suppose that $\mathcal{T}Z \xrightarrow{z} Z, \epsilon : z.\eta \Rightarrow 1_Z$ is a left-semi $\mathcal{T}$-algebra, so that both $z.\eta$ and 1_Z are strict endomaps. Then $\epsilon : z.\eta \Rightarrow 1_Z$ is a left-semi $\mathcal{T}$-algebra 2-cell.*

At this point, it is straightforward to check that left-semi $\mathcal{T}$-algebras, strict maps and 2-cells forms a 2-category that we denote as $\mathbf{ls}\text{-}\mathcal{T}\text{-}\mathbf{Alg}_s$.

Looking more closely at what we showed above we see that if we set $f = z.\eta$, then we have $f = f^2$ and $\epsilon.f = \mathrm{id}_f = f.\epsilon$. So in fact we have the following.

▶ **Proposition 9.** *Suppose that $\mathcal{T}Z \xrightarrow{z} Z, \epsilon : z.\eta \Rightarrow 1_Z$ is a left-semi $\mathcal{T}$-algebra. Then, in the 2-category $\mathbf{ls}\text{-}\mathcal{T}\text{-}\mathbf{Alg}_s$, the 1-cell f and the 2-cell $\epsilon : f \Rightarrow 1_Z$ equips the left-semi $\mathcal{T}$-algebra with the structure of a strictly idempotent comonad.*

Applying the evident forgetful 2-functor we get that $f = f^2$ and $\epsilon : f \Rightarrow 1_Z$ equips Z with the structure of a strictly idempotent comonad in the underlying 2-category $\mathcal{K}$.

▶ **Proposition 10.** *Suppose that $\mathcal{T}X \xrightarrow{x} X$ is a $\mathcal{T}$-algebra and $f = f^2 : X \to X$ and $\epsilon : f \Rightarrow 1_X$ equips X with the structure of a strictly idempotent comonad natural in $\mathcal{T}\text{-}\mathbf{Alg}_s$. Then $\mathcal{T}X \xrightarrow{x} X \xrightarrow{f} X, \epsilon : fx.\eta \Rightarrow 1_X$ is a left-semi $\mathcal{T}$-algebra.*

Proof sketch. The 1-cell part is routine and the 2-cell uses that ϵ is a 2-cell in $\mathcal{T}\text{-}\mathbf{Alg}_s$. ◀

▶ **Definition 11.** *Suppose that $\mathcal{S}$ and $\mathcal{T}$ are 2-monads. A left-semi monad map from the first to the second consists of $\lambda : \mathcal{S} \to \mathcal{T}$ satisfying the following equalities*

$$\begin{tikzcd} 1 \arrow[r, "\eta"] \arrow[dr, "\eta"'] & \mathcal{S} \arrow[d, "\lambda"] \arrow[dl, phantom, "\Leftarrow\gamma"] \\ & \mathcal{T} \end{tikzcd} \quad (6) \qquad \begin{tikzcd} \mathcal{S}^2 \arrow[r, "\lambda\mathcal{S}"] \arrow[d, "\mu"'] & \mathcal{T}\mathcal{S} \arrow[r, "\mathcal{T}\lambda"] & \mathcal{T}^2 \arrow[d, "\mu"] \\ \mathcal{S} \arrow[rr, "\lambda"'] & & \mathcal{T} \end{tikzcd} \text{ or equivalently } \begin{tikzcd} \mathcal{S}^2 \arrow[r, "\mathcal{S}\lambda"] \arrow[d, "\mu"'] & \mathcal{S}\mathcal{T} \arrow[r, "\lambda\mathcal{T}"] & \mathcal{T}^2 \arrow[d, "\mu"] \\ \mathcal{S} \arrow[rr, "\lambda"'] & & \mathcal{T} \end{tikzcd} \quad (7)$$

$$\begin{tikzcd} \mathcal{S} \arrow[r, "\mathcal{S}\eta"] \arrow[dr, "\mathcal{S}\eta"'] & \mathcal{S}^2 \arrow[d, "\mathcal{S}\lambda"] \arrow[dl, phantom, "\Leftarrow\mathcal{S}\gamma"] & \\ & \mathcal{S}\mathcal{T} \arrow[d, "\lambda\mathcal{T}"] & \\ & \mathcal{T}^2 \arrow[r, "\mu"] & \mathcal{T} \end{tikzcd} = \begin{tikzcd} \mathcal{S} \arrow[d, bend right, "\lambda"'] \arrow[d, bend left, "\lambda"] \arrow[d, phantom, "="] \\ \mathcal{T} \end{tikzcd} = \begin{tikzcd} \mathcal{S} \arrow[d, "\lambda"] & & \\ \mathcal{T} \arrow[r, "\eta\mathcal{T}"] \arrow[dr, "\eta\mathcal{T}"'] & \mathcal{S}\mathcal{T} \arrow[d, "\lambda\mathcal{T}"] \arrow[dl, phantom, "\Leftarrow\gamma\mathcal{T}"] & \\ & \mathcal{T}^2 \arrow[r, "\mu"] & \mathcal{T} \end{tikzcd} \quad (8)$$

$$
\begin{tikzcd} A \arrow[d, "\lambda"'] \arrow[dr, bend left, "k"] \arrow[dr, phantom, "\Nearrow\alpha"'] & \\ B \arrow[r, "\ell"'] & C \end{tikzcd}
\qquad
\begin{tikzcd} A \arrow[d, "\lambda"'] \arrow[dr, bend left, "f"] \arrow[dr, phantom, "\Nearrow\phi"'] & \\ B \arrow[r, "g"'] & D \end{tikzcd}
\qquad
\begin{tikzcd} A \arrow[d, "\lambda"'] \arrow[dr, bend left=50, "f'"] \arrow[dr, bend right=10, "f"'] \arrow[dr, phantom, "\Nearrow\rho"] \arrow[d, phantom, shift left=4, "\phi\Nearrow"] & \\ B \arrow[r, "g"'] & D \end{tikzcd}
=
\begin{tikzcd} A \arrow[d, "\lambda"'] \arrow[dr, bend left=50, "f'"] \arrow[d, phantom, shift left=4, "\phi'\Nearrow"] & \\ B \arrow[r, "g"'] \arrow[r, bend left, "g'"] \arrow[r, phantom, shift left=2, "\sigma\Uparrow"] & D \end{tikzcd}
$$

Figure 1 Cocones under the arrow λ.

136 ▶ **Proposition 12.** *Suppose that* $\mathcal{T}Z \xrightarrow{z} Z, \epsilon : z.\eta \Rightarrow 1_Z$ *is a left-semi* $\mathcal{T}$*-algebra and* $\mathcal{S} \xrightarrow{\lambda} \mathcal{T}, \gamma : \lambda.\eta \Rightarrow \eta$ *is a left-semi monad map. Then* $\mathcal{S}Z \xrightarrow{\lambda_Z} \mathcal{T}Z \xrightarrow{z} Z, \epsilon.\gamma : z.\lambda.\eta \Rightarrow 1_Z$ *is a left-semi* $\mathcal{S}$*-algebra.*

Proof sketch. The 1-cell part is routine and the 2-cell parts use the naturality of λ to separate the two 2-cells γ and ϵ. ◀

1.3 Colax colimits induced by a map in 2-category

In this section we review the notion of colax colimits in a cocomplete 2-category specialised to our context [3, 20].

In the 2-category $\mathcal{K}$, suppose that α is a colax cocone (k, ℓ, α) under the arrow λ (see Figure 1, left). Then, for every D, composition with α induces an isomorphism of categories between $\mathcal{K}(C, D)$ and the category of colax cocones under the arrow λ with objects (f, g, ϕ) (see Figure 1, center) and 1-cells $(f, g, \phi) \to (f', g', \phi')$ given by 2-cells $f \overset{\rho}{\Rightarrow} f'$ and $g \overset{\sigma}{\Rightarrow} g'$ such that $\rho * \phi = \phi' * \sigma.\lambda$ (see Figure 1, right).

This isomorphism of categories has two universal aspects, the first is 1-dimensional and the second is 2-dimensional:

- for any $\begin{tikzcd} A \arrow[d, "\lambda"'] \arrow[dr, bend left, "f"] \arrow[dr, phantom, "\Nearrow\phi"'] & \\ B \arrow[r, "g"'] & D \end{tikzcd}$ there is a unique r such that $\begin{tikzcd} A \arrow[d, "\lambda"'] \arrow[dr, bend left, "k"] \arrow[dr, phantom, "\Nearrow\alpha"'] & & \\ B \arrow[r, "\ell"'] & C \arrow[r, "r"] & D \end{tikzcd} = \phi$
- for any $\begin{tikzcd} A \arrow[d, "\lambda"'] \arrow[dr, bend left=50, "f'"] \arrow[dr, bend right=10, "f"'] \arrow[dr, phantom, "\Nearrow\rho"] \arrow[d, phantom, shift left=4, "\phi\Nearrow"] & \\ B \arrow[r, "g"'] & D \end{tikzcd} = \begin{tikzcd} A \arrow[d, "\lambda"'] \arrow[dr, bend left=50, "f'"] \arrow[d, phantom, shift left=4, "\phi'\Nearrow"] & \\ B \arrow[r, "g"'] \arrow[r, bend left, "g'"] \arrow[r, phantom, shift left=2, "\sigma\Uparrow"] & D \end{tikzcd}$ there is a unique $r \overset{\tau}{\Rightarrow} r'$ such that

$$
\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f"'] \arrow[r, phantom, "\rho\Uparrow"] & D \end{tikzcd}
=
\begin{tikzcd} A \arrow[r, "k"] & {} \arrow[r, bend left, "r'"] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\tau\Uparrow"] & D \end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd} B \arrow[r, bend left, "g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\sigma\Uparrow"] & D \end{tikzcd}
=
\begin{tikzcd} B \arrow[r, "\ell"] & C \arrow[r, bend left, "r'"] \arrow[r, bend right, "r"'] \arrow[r, phantom, "\tau\Uparrow"] & D \end{tikzcd}
\tag{9}
$$

Although we will compute colax colimits in the 2-category of $\mathcal{L}$-$\mathbf{Alg}_s$ where what happens is more subtle, we illustrate this definition by computing colax colimits in the 2-category **Cat**.

▶ **Example 13.** In **Cat**, $A \xrightarrow{\lambda} B$ is a functor between categories. The colax colimit under λ is a category C which consists of separate copies of A and B together with, for every object $a \in A$, new maps $\lambda(a) \xrightarrow{\alpha_a} a$, composition of such and evident identifications. Precisely, maps from $b \in B$ to $a \in A$ are given by $b \xrightarrow{v} \lambda(a) \xrightarrow{\alpha_A} a$ and $C(b, a) \simeq B(b, \lambda(a))$.

2 The colimit 2-monad induced by a map of 2-monads

From now on, we assume that $\mathcal{L}$ is a finitary 2-monad, so that $\mathcal{L}$-$\mathbf{Alg}_s$ is cocomplete [19].

2.1 Definition of the colimit and its 2-naturality

▶ **Proposition 14.** *Suppose that $\lambda : \mathcal{L} \to \mathcal{M}$ is a map of 2-monads. Then the colax colimit $(\mathcal{Q}X, u)$ under the induced $\lambda_X : (\mathcal{L}X, \mu^{\mathcal{L}}) \to (\mathcal{M}X, \mu^{\mathcal{M}})$ in $\mathcal{L}$-$\mathbf{Alg}_s$ is natural in $(\mathcal{L}X, \mu^{\mathcal{L}})$*

$$\begin{tikzcd} \mathcal{L}X \arrow[d, "\lambda"'] \arrow[dr, bend left, "k"] & \\ \mathcal{M}X \arrow[r, "\ell"'] \arrow[ur, phantom, "\Nearrow\alpha"] & \mathcal{Q}X \end{tikzcd} \tag{10}$$

Proof sketch. Assume $\begin{tikzcd} \mathcal{L}A \arrow[r, bend left, "g'"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Uparrow"] & \mathcal{L}B \end{tikzcd}$ is an $\mathcal{L}$-algebra 2-cell. For each 1-cell we get by 2-cell naturality a cocone and so we get a unique maps $\widehat{g}$ and $\widehat{g'}$ mapping $\mathcal{Q}A$ to $\mathcal{Q}B$ arising from 1-cell universality. We then have

$$\begin{tikzcd} \mathcal{L}A \arrow[r, "g"] & \mathcal{L}B \arrow[d, "\lambda"] \arrow[dr, bend left, "k"] & \\ & \mathcal{M}B \arrow[r, "\ell"'] \arrow[ur, phantom, "\Nearrow\alpha"] & \mathcal{Q}B \end{tikzcd} = \begin{tikzcd} \mathcal{L}A \arrow[d, "\lambda"'] \arrow[dr, bend left, "k"] & & \\ \mathcal{M}A \arrow[r, "\ell"'] \arrow[ur, phantom, "\Nearrow\alpha"] & \mathcal{Q}A \arrow[r, "\widehat{g}"] & \mathcal{Q}B \end{tikzcd}$$

and similarly for g' and $\widehat{g'}$. By 2-cell universality (9), we then get:

$$\begin{tikzcd} \mathcal{L}A \arrow[r, bend left, "g'"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Uparrow"] & \mathcal{L}B \arrow[r, "k"] & \mathcal{Q}B \end{tikzcd} = \begin{tikzcd} \mathcal{L}A \arrow[r, "k"'] & \mathcal{Q}A \arrow[r, bend left, "\widehat{g'}"] \arrow[r, bend right, "\widehat{g}"'] \arrow[r, phantom, "\Uparrow"] & \mathcal{Q}B \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{M}A \arrow[r, bend left, "\lambda^*\lambda_! g'"] \arrow[r, bend right, "\lambda^*\lambda_! g"'] \arrow[r, phantom, "\Uparrow"] & \mathcal{M}B \arrow[r, "\ell"] & \mathcal{Q}B \end{tikzcd} = \begin{tikzcd} \mathcal{M}A \arrow[r, "\ell"'] & \mathcal{Q}A \arrow[r, bend left, "\widehat{g'}"] \arrow[r, bend right, "\widehat{g}"'] \arrow[r, phantom, "\Uparrow"] & \mathcal{Q}B \end{tikzcd}$$

◀

2.2 A left semi-algebra

We explore the properties of $\mathcal{Q}X$ by considering 1 and 2 dimensional aspects of trivial cocones under λ. From the identity cocone under λ, a unique $\mathcal{L}$-algebra map h arises by 1-dimensional universality.

$$\begin{tikzcd} \mathcal{L}X \arrow[d, "\lambda"'] \arrow[dr, bend left, "\lambda"] & \\ \mathcal{M}X \arrow[r, equal, "1"'] \arrow[ur, phantom, "="] & \mathcal{M}X \end{tikzcd} = \begin{tikzcd} \mathcal{L}X \arrow[d, "\lambda"'] \arrow[dr, bend left, "k"] & & \\ \mathcal{M}X \arrow[r, "\ell"'] \arrow[ur, phantom, "\Nearrow\alpha"] & \mathcal{Q}X \arrow[r, "h"] & \mathcal{M}X \end{tikzcd} \quad \text{and} \quad \begin{cases} h\,k = \lambda_X \\ h\,\ell = 1_{\mathcal{M}X} \\ h.\alpha = \mathrm{id}_\lambda \end{cases} \tag{11}$$

If $\begin{tikzcd} \mathcal{L}A \arrow[r, bend left, "g'"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Uparrow"] & \mathcal{L}B \end{tikzcd}$ is an $\mathcal{L}$-algebra 2-cell, then by 2-dimensional universality, so h is natural

$$\begin{tikzcd} \mathcal{Q}A \arrow[r, "h"] & \mathcal{M}A \arrow[r, bend left, "\lambda^*\lambda_! g'"] \arrow[r, bend right, "\lambda^*\lambda_! g"'] \arrow[r, phantom, "\Uparrow"] & \mathcal{M}B \end{tikzcd} = \begin{tikzcd} \mathcal{Q}A \arrow[r, bend left, "\widehat{g'}"] \arrow[r, bend right, "\widehat{g}"'] \arrow[r, phantom, "\Uparrow"] & \mathcal{Q}B \arrow[r, "h"] & \mathcal{M}B \end{tikzcd}.$$

From the 2-cells $\mathrm{id}_\ell : \ell = \ell$ and $\alpha : \ell\,\lambda \Rightarrow k$, arises a unique $\mathcal{L}$-$\mathbf{Alg}_s$ 2-cell $\beta : \ell\,h \Rightarrow 1_{\mathcal{Q}X}$ s.t.

$$\begin{tikzcd} \mathcal{L}X \arrow[d, "k"] & & \\ \mathcal{Q}X \arrow[r, "h"] \arrow[rr, bend left=50, equal, "1_{\mathcal{Q}X}"] \arrow[rr, phantom, bend left=20, "\beta\Uparrow"] & \mathcal{M}X \arrow[r, "\ell"] & \mathcal{Q}X \end{tikzcd} = \begin{tikzcd} \mathcal{L}X \arrow[d, "h"] \arrow[dr, bend left, "k"] & \\ \mathcal{M}X \arrow[r, "\ell"'] \arrow[ur, phantom, "\alpha\Nearrow"] & \mathcal{Q}X \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{M}X \arrow[d, "\ell"] & & \\ \mathcal{Q}X \arrow[r, "h"] \arrow[rr, bend left=50, equal, "1_{\mathcal{Q}X}"] \arrow[rr, phantom, bend left=20, "\beta\Uparrow"] & \mathcal{M}X \arrow[r, "\ell"] & \mathcal{Q}X \end{tikzcd} = \begin{tikzcd} \mathcal{M}X \arrow[d, bend right, "\ell"'] \arrow[d, bend left, "\ell"] \arrow[d, phantom, "="] \\ \mathcal{Q}X \end{tikzcd} \tag{12}$$

Denote $f = \ell h$. Then $\mathcal{Q}X$ is a $\mathcal{L}$-algebra and $f = f^2 : \mathcal{Q}X \to \mathcal{Q}X$ and $\beta : f \Rightarrow 1_{\mathcal{Q}X}$ equips $\mathcal{Q}X$ with the structure of a strictly idempotent comonad natural in $\mathcal{L}$-$\mathbf{Alg}_s$ as $\beta.\ell = \mathrm{id}_\ell$, $\beta.k = \alpha$, and thus $h.\beta = \mathrm{id}_\ell$. We apply Proposition 10 and get

▶ **Proposition 15.** *$\mathcal{L}\mathcal{Q}X \xrightarrow{u} \mathcal{Q}X \xrightarrow{h} \mathcal{M}X \xrightarrow{\ell} \mathcal{Q}X$ with $\beta : \ell h u \eta^{\mathcal{L}} = \ell h \Rightarrow 1_{\mathcal{Q}X}$ is a left-semi $\mathcal{L}$-algebra.*

▶ **Proposition 16.** *Assume z denotes the map $\mathcal{M}\mathcal{Q}X \xrightarrow{\mathcal{M}h} \mathcal{M}^2X \xrightarrow{\mu^{\mathcal{M}}} \mathcal{M}X \xrightarrow{\ell} \mathcal{Q}X$. Then $\mathcal{Q}X$ together with z and $z\eta^{\mathcal{M}} = \ell h \overset{\beta}{\Rightarrow} 1_{\mathcal{Q}X}$ is a left-semi $\mathcal{M}$-algebra.*

Proof sketch. The 2-cell property relies on $\beta.\ell = \mathrm{id}_\ell$ and $h.\beta = \mathrm{id}_h$. ◀

As λ is a map of 2-monads, it is a left-semi monad map and we apply Proposition 12 and get

▶ **Proposition 17.** *$\mathcal{L}\mathcal{Q}X \xrightarrow{\lambda\mathcal{Q}} \mathcal{M}\mathcal{Q}X \xrightarrow{\mathcal{M}\ell} \mathcal{M}^2X \xrightarrow{\mu^{\mathcal{M}}} \mathcal{M}X \xrightarrow{\ell} \mathcal{Q}X$ together with the 2-cell $\beta : z\,(\lambda\mathcal{Q})\eta^{\mathcal{L}} = \ell h \Rightarrow 1_{\mathcal{Q}X}$ is a left-semi $\mathcal{L}$-algebra.*

The following is an immediate consequence of the definitions.

▶ **Proposition 18.** *The left-semi $\mathcal{L}$-algebras of Proposition 15 and 17 are equal.*

Let us recap the properties of $\mathcal{Q}X$. It is equipped with an $\mathcal{L}$-algebra structure u and a left-semi $\mathcal{M}$-algebra structure z whose 2-cell β lies in $\mathcal{L}$-$\mathbf{Alg}_s$ and such that the two resulting left-semi $\mathcal{L}$-algebra structure coincide.

In order to prove that $\mathcal{Q}$ is a 2-monad (Theorem 22) and that these properties characterise $\mathcal{Q}$-algebras (Theorem 25), we introduce an eccentric lemma. Given this structure on a general object X, we can build a map $\mathcal{Q}X \to X$ in a sufficiently functorial way that both theorems follow. What we need is the 1-cell and 2-cell aspects associated to these properties.

2.3 The Structure category

Let us define the Structure category $\mathfrak{Q}$

- an object of $\mathfrak{Q}$ consists of an object X of $\mathbf{K}$ equipped with
  - the structure $\mathcal{L}X \xrightarrow{w} X$ of an $\mathcal{L}$-algebra
  - the structure $\mathcal{M}X \xrightarrow{u} X$, $\epsilon : z\,\eta^{\mathcal{M}} = f \Rightarrow 1_X$ of a left-semi $\mathcal{M}$-algebra

  such that
  - f is an endomap of the $\mathcal{L}$-algebra $\mathcal{L}X \xrightarrow{w} X$ and ϵ is an $\mathcal{L}$-algebra 2-cell
  - the two induced left-semi $\mathcal{L}$-algebra structures, with structure maps $\mathcal{L}X \xrightarrow{w} X \xrightarrow{\eta^{\mathcal{M}}} X \xrightarrow{f} X$ and $\mathcal{L}X \xrightarrow{\lambda} \mathcal{M}X \xrightarrow{z} X$, are equal
- a map in $\mathfrak{Q}$ between objects X and X' equipped as above is a map $p : X \to X'$ in $\mathbf{K}$ which is both an $\mathcal{L}$-algebra and a left-semi $\mathcal{M}$-algebra map
- a 2-cell between two such maps p and q is a 2-cell $p \Rightarrow p'$ which is both an $\mathcal{L}$-algebra and a left-semi $\mathcal{M}$-algebra 2-cell.

▶ Remark 19. **1.** In the definition, the condition regarding the left-semi $\mathcal{L}$-algebra structures amounts to the claim that $f\,w = z\,\lambda$. The equality of the 2-cells is then automatic

2. It is a consequence of the definition that $z : \mathcal{M}X \to X$ is a map of $\mathcal{L}$-algebras. Indeed, if we consider the three following conditions, any two of them implies the third.
  - f is an endomap of $\mathcal{L}$-algebras,
  - $f\,w = \lambda\,z$
  - z is a map of $\mathcal{L}$-algebras

▶ **Proposition 20.** *$\mathcal{Q}X$ together with u, z and α is an object in $\mathfrak{Q}$.*

Assume X together with w, z, and ϵ is an object in $\mathfrak{Q}$. Then we define $\mathcal{Q}X \xrightarrow{x} X$ to be the unique $\mathcal{L}$-$\mathbf{Alg}_s$ map arising from the colax cocone

$$
\begin{tikzcd}
\mathcal{L}X \arrow[r, "w"] \arrow[d, "\lambda"'] & X \arrow[d, equal] \\
\mathcal{M}X \arrow[r, "z"'] \arrow[ur, bend left, "f"] \arrow[ur, phantom, "\overset{\epsilon}{\Rightarrow}"] & X
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\mathcal{L}X \arrow[d, "\lambda"'] \arrow[dr, bend left, "k"] & & \\
\mathcal{M}X \arrow[r, "\ell"'] \arrow[ur, phantom, "\overset{\alpha}{\Rightarrow}"] & \mathcal{Q}X \arrow[r, "x"'] & X
\end{tikzcd}
\tag{13}
$$

▶ **Proposition 21.** *Assume X together with w, z, and ϵ is an object in $\mathfrak{Q}$ and x denotes the associated map. Then $x : \mathcal{Q}X \to X$ is a map in $\mathfrak{Q}$ which is natural in X.*

Sketch proof. Assume X' together with w', z', ϵ' in $\mathfrak{Q}$ associated with x' and $p \overset{\rho}{\Rightarrow} q$ a 2-cell in $\mathfrak{Q}$. Then $\begin{tikzcd} \mathcal{Q}X' \arrow[r, bend left, "\mathcal{Q}p"] \arrow[r, bend right, "\mathcal{Q}q"'] \arrow[r, phantom, "\Downarrow \mathcal{Q}\rho"] & \mathcal{Q}X \arrow[r, "x"] & X \end{tikzcd} = \begin{tikzcd} \mathcal{Q}X' \arrow[r, "x'"] & X' \arrow[r, bend left, "p"] \arrow[r, bend right, "q"'] \arrow[r, phantom, "\Downarrow\rho"] & X \end{tikzcd}$ by 2-cell universality. ◀

2.4 The colimit is a monad

As $\mathcal{Q}X$ is an object in $\mathfrak{Q}$ (Proposition 20), the induced map $\mathcal{Q}^2X \xrightarrow{\mu^{\mathcal{Q}}} \mathcal{Q}X$ is a map in $\mathfrak{Q}$ (Proposition 21).

Assume (X, w, z, ϵ) in $\mathfrak{Q}$. Then the induced map $\mathcal{Q}X \xrightarrow{x} \mathcal{Q}X$ is a map in $\mathfrak{Q}$. We apply the 1-cell part of the naturality (Proposition 21) with $p = x$ and $x' = \mu^{\mathcal{Q}}$ and get

$$
\begin{tikzcd}
\mathcal{Q}^2X \arrow[r, "\mu^{\mathcal{Q}}"] \arrow[d, "\mathcal{Q}x"'] & \mathcal{Q}X \arrow[d, "x"] \\
\mathcal{Q}X \arrow[r, "x"'] & X
\end{tikzcd}
\quad \text{in particular, setting } x = \mu^{\mathcal{Q}} \quad
\begin{tikzcd}
\mathcal{Q}^3X \arrow[r, "\mu^{\mathcal{Q}}"] \arrow[d, "\mathcal{Q}\mu^{\mathcal{Q}}"'] & \mathcal{Q}X \arrow[d, "\mu^{\mathcal{Q}}"] \\
\mathcal{Q}^2X \arrow[r, "\mu^{\mathcal{Q}}"'] & \mathcal{Q}X
\end{tikzcd}
$$

▶ **Theorem 22.** *$\mathcal{Q}$ is a 2-monad with multiplication $\mu^{\mathcal{Q}}$ and unit $X \xrightarrow{\eta^{\mathcal{L}}} \mathcal{L}X \xrightarrow{k} \mathcal{Q}X$.*

▶ **Proposition 23.** *$\mathcal{L} \xrightarrow{k} \mathcal{Q}$ is a map of monads.*

Proof sketch. The unit aspect is by definition of $\eta^{\mathcal{Q}}$. As k is a map of $\mathcal{L}$-algebra and $\mu^{\mathcal{Q}}\, k = u$ by cocone equality (13), we get the multiplication diagram. ◀

▶ **Proposition 24.** *$\mathcal{M} \xrightarrow{\ell} \mathcal{Q}$ is a left-semi map of monads.*

Proof sketch. Recall that $h\ell = 1$ and that $\mu^{\mathcal{Q}}\,(\ell\mathcal{Q}) = z$ by cocone equality (13). Then, the multiplication diagram (8) follows since $\mu^{\mathcal{Q}}\,(\ell\mathcal{Q})\,(\mathcal{L}\ell) = z\,(\mathcal{L}\ell) = \ell\,\mu^{\mathcal{M}}\,(\mathcal{M}h)\,(\mathcal{M}\ell) = \ell\,\mu^{\mathcal{M}}$.

We define the unit 2-cell $\gamma : \ell\,\eta^{\mathcal{M}} \Rightarrow \eta^{\mathcal{Q}}$ in (6) as

$$
\begin{tikzcd}
X \arrow[r, "\eta^{\mathcal{M}}"] \arrow[dr, "\eta^{\mathcal{L}}"'] & \mathcal{M}X \arrow[dr, "\ell"] & \\
& \mathcal{L}X \arrow[u, "\lambda"] \arrow[r, "k"'] \arrow[ur, phantom, "\overset{\alpha}{\Rightarrow}"] & \mathcal{Q}X
\end{tikzcd}
$$

We prove Equalities (7). Recall that $\alpha = \beta.k$ and $\beta.\ell = \mathrm{id}_\ell$. As $\mu^{\mathcal{Q}}\,(\ell\mathcal{Q}) = z = \ell\,\mu^{\mathcal{M}}\,(\mathcal{M}h)$ and $h.\alpha = h.\beta.k = \mathrm{id}_\ell\,.k$

$$\begin{tikzcd}
\mathcal{M}X \arrow[r, "\mathcal{M}\eta^{\mathcal{M}}"] \arrow[dr, "\mathcal{M}\eta^{\mathcal{L}}"'] & \mathcal{M}^2X \arrow[dr, "\mathcal{M}\ell"] & & & \\
& \mathcal{M}\mathcal{L}X \arrow[u, "\mathcal{M}\lambda"] \arrow[r, "\mathcal{M}k"'] \arrow[r, phantom, "\mathcal{M}\alpha\Downarrow", bend left] & \mathcal{M}\mathcal{Q}X \arrow[r, "\mathcal{M}h"] \arrow[d, "\ell\mathcal{Q}"] & \mathcal{M}^2X \arrow[r, "\mu^{\mathcal{M}}"] & \mathcal{M}X \arrow[d, "\ell"] \\
& & \mathcal{Q}^2X \arrow[rr, "\mu^{\mathcal{Q}}"] & & \mathcal{Q}X
\end{tikzcd} = \mathrm{id}_\ell\,.$$

As $\mu^{\mathcal{Q}}.\alpha = \beta.u$ (see Equality (13) with $x = \mu^{\mathcal{Q}}$), and as u is an $\mathcal{L}$-algebra $u\,(\eta^{\mathcal{L}}\mathcal{Q}) = 1_{\mathcal{Q}X}$ so the second 2-cell equality follows: $\mu^{\mathcal{Q}}.\alpha.(\eta^{\mathcal{L}}\mathcal{Q})\,\ell = \beta.u\,(\eta^{\mathcal{L}}\mathcal{Q})\,\ell = \beta.\ell = \mathrm{id}_\ell$. ◀

▶ **Theorem 25.** *The 2-category $\mathcal{Q}$-$\mathbf{Alg}_s$ of the 2-monad $\mathcal{Q}$ is isomorphic to the Structure category.*

Proof sketch. It remains to prove the direct implication. Assume $\mathcal{Q}X \xrightarrow{x} X$ is a $\mathcal{Q}$-algebra.

- Since $k : \mathcal{L} \to \mathcal{Q}$ is a monad map, $w : \mathcal{L}X \xrightarrow{k} \mathcal{Q}X \xrightarrow{x} X$ is an $\mathcal{L}$-algebra.
- By Propositions 12, since $\ell : \mathcal{M} \to \mathcal{Q}$ is a left-semi monad map, $z : \mathcal{M}X \xrightarrow{\ell} \mathcal{Q}X \xrightarrow{x} X$ is a left-semi $\mathcal{M}$-algebra with 2-cell α where we denote $f_x = z\,\eta^{\mathcal{M}}$

$$\begin{tikzcd}
X \arrow[r, "f_x"] \arrow[r, phantom, "\Downarrow\epsilon", bend right=40] \arrow[r, equal, bend right=60] & X
\end{tikzcd}
=
\begin{tikzcd}
X \arrow[r, "\eta^{\mathcal{M}}"] \arrow[dr, "\eta^{\mathcal{L}}"'] & \mathcal{M}X \arrow[r, "\ell"] \arrow[dr, phantom, "\alpha\Downarrow"] & \mathcal{Q}X \arrow[r, "x"] & X \\
& \mathcal{L}X \arrow[u, "\lambda"] \arrow[ur, "k"'] & &
\end{tikzcd} \tag{14}$$

- We know that $h\,\ell = \lambda$ and $h\ell = 1_{\mathcal{Q}X}$ and $z = x\,\ell$ is a left-semi $\mathcal{M}$-algebra. We deduce $\mathcal{L}X \xrightarrow{w} X \xrightarrow{f}_x X = \mathcal{L}X \xrightarrow{\lambda} \mathcal{M}X \xrightarrow{z} X$ using the following.

$$\begin{tikzcd}
\mathcal{L}X \arrow[rr, "k"] \arrow[dr, "\lambda"] \arrow[dddd, "\lambda"'] & & \mathcal{Q}X \arrow[rr, "x"] \arrow[d, "\eta^{\mathcal{M}}\mathcal{Q}"] \arrow[dl, "h"'] & & X \arrow[dd, "\eta^{\mathcal{M}}"] \\
& & \mathcal{M}\mathcal{Q}X \arrow[d, "\mathcal{M}h"] \arrow[dr, equal] & & \\
& \mathcal{M}X \arrow[r, "\eta^{\mathcal{M}}\mathcal{M}"] \arrow[dr, equal] & \mathcal{M}^2X \arrow[r, "\mathcal{M}\ell"'] \arrow[d, "\mu^{\mathcal{M}}"] & \mathcal{M}\mathcal{Q}X \arrow[r, "\mathcal{M}x"] & \mathcal{M}X \arrow[d, "\ell"] \\
& & \mathcal{M}X \arrow[d, "\ell"] & & \mathcal{Q}X \arrow[d, "x"] \\
\mathcal{M}X \arrow[rr, "\ell"] & & \mathcal{Q}X \arrow[rr, "x"] & & X
\end{tikzcd}$$

- We prove that ϵ is in $\mathcal{L}$-$\mathbf{Alg}_s$. We first remark that $x.\beta = \epsilon.x$. Indeed, by naturality of $\eta^{\mathcal{M}}$ and of α, we have $\alpha.\eta^{\mathcal{L}}\,x = (\mathcal{Q}x).\alpha.\eta^{\mathcal{L}}$. Because x is a $\mathcal{Q}$-algebra, $x.\alpha.\eta^{\mathcal{L}}\,x = x\,(\mathcal{Q}x).\alpha.\eta^{\mathcal{L}} = x\,\mu^{\mathcal{Q}}.\alpha.\eta^{\mathcal{L}}$ and we conclude as $\mu^{\mathcal{Q}}.\alpha.\eta^{\mathcal{L}} = \beta$.
  Then, as β is an $\mathcal{L}$-algebra 2-cell by construction and x is a $\mathcal{L}$-algebra, so that $\epsilon.x$ is a $\mathcal{L}$-algebra 2-cell. This can be represented by the lhs 2-cell equality which results in the

rhs equality by precomposition by $\mathcal{L}\eta^{\mathcal{Q}}$

$$
\begin{tikzcd}
\mathcal{L}\mathcal{Q}X \arrow[r, "\mathcal{L}x"] \arrow[d, "u"] & \mathcal{L}X \arrow[r, "\mathcal{L}f_x"] \arrow[r, bend right=60, equal, ""'] \arrow[r, phantom, shift right=3, "\Downarrow\mathcal{L}\epsilon"] & \mathcal{L}X \arrow[d, "w"] \\
\mathcal{Q}X \arrow[r, "x"] & X \arrow[r, "f_x"] \arrow[r, bend right=60, equal] \arrow[r, phantom, shift right=3, "\Downarrow\epsilon"] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{L}X \arrow[r, "\mathcal{L}f_x"] \arrow[r, bend right=60, equal] \arrow[r, phantom, shift right=3, "\Downarrow\mathcal{L}\epsilon"] \arrow[d, "w"] & \mathcal{L}X \arrow[d, "w"] \\
X \arrow[r, "f_x"] \arrow[r, bend right=60, equal] \arrow[r, phantom, shift right=3, "\Downarrow\epsilon"] & X
\end{tikzcd}
$$

This proves that ϵ is an $\mathcal{L}$-algebra 2-cell.

◀

Our analysis of the 2-monad $\mathcal{Q}$ involved consideration of left-semi $\mathcal{M}$-algebras. We can immediately say something about them. Suppose that $\mathcal{M}^+$ is the result of applying our construction to the map $\eta : \mathcal{I} \to \mathcal{M}$ of monads given by the unit. By Theorem 25, we deduce the following.

▶ **Proposition 26.** *$\mathcal{M}^+$-$\mathbf{Alg}_s$ is isomorphic to* $\mathbf{ls}$-$\mathcal{M}$-$\mathbf{Alg}_s$

So the 2-category of left-semi $\mathcal{M}$-algebras is in fact monadic over the base $\mathcal{K}$.

3 The Linear-non-linear 2-monad

In this section, we show how our theory applies in the case of most immediate interest to us. We take for $\mathcal{L}$ the 2-monad for symmetric strict monoidal categories: we give a concrete presentation in 3.1. We take for $\mathcal{M}$ the 2-monad for categories with strict finite products: we give a concrete presentation in 3.2. There is an evident map of monads $\mathcal{L} \to \mathcal{M}$ and in 3.3, we describe the 2-monad $\mathcal{Q}$ obtained by our construction.

In further work we shall develop general theory to show that this $\mathcal{Q}$ in particular extends from **CAT** to profunctors. This gives a notion of algebraic theory in the sense of Hyland [16] and we shall use that to handle the linear and non-linear substitutions appearing in differential lambda-calculus [8].

3.1 The 2-monad for symmetric strict monoidal categories

For a category A, let $\mathcal{L}A$ be the following category. The objects are finite sequences $\langle a_i \rangle_{i\in[n]}$ with $n \in \mathbf{N}$ and $a_i \in A$. The morphisms

$$\langle a_i \rangle_{i\in[n]} \to \langle a'_j \rangle_{j\in[m]}$$

consist of a bijection $\sigma : [n] \to [m]$ (so n and m are equal) and for each $j \in [m]$ a map $a_{\sigma(j)} \to b_j$ in A. The identity and composition are evident.

$\mathcal{L}$ extends readily to a 2-functor on **CAT** and it has the structure of a 2-monad where $\eta^{\mathcal{L}} : A \to \mathcal{L}A$ takes a to the singleton $\langle a \rangle$ and $\mu^{\mathcal{L}} : \mathcal{L}^2 A \to \mathcal{L}A$ acts on objects by concatenation of sequences.

Each $\mathcal{L}A$ has the structure of a symmetric empty sequence and tensor product is given by concatenation. One can check directly that $A \xrightarrow{\eta^{\mathcal{L}}} \mathcal{L}A$ makes $\mathcal{L}A$ the free symmetric strict monoidal category on A. Moreover to equip A with the structure of a symmetric strict monoidal category is to give A an $\mathcal{L}$-algebra structure. Maps and 2-cells are as expected so we identify $\mathcal{L}$-$\mathbf{Alg}_s$ as the 2-category of strict monoidal categories, strict monoidal functors and monoidal 2-cells.

3.2 The 2-monad for categories with products

For a category A, let $\mathcal{M}A$ be the following category. The objects are finite sequences $\langle a_i \rangle_{i\in[n]}$ with $n \in \mathbf{N}$ and $a_i \in A$. The morphisms

$$\langle a_i \rangle_{i\in[n]} \to \langle a'_j \rangle_{j\in[m]}$$

consist of a map $\phi : [m] \to [n]$ and for each $j \in [m]$ a map $a_{\phi(j)} \to b_j$ in A. The identity and composition are evident.

$\mathcal{M}$ extends readily to a 2-functor on **CAT** and it has the structure of a 2-monad where $\eta^{\mathcal{M}} : A \to \mathcal{L}A$ takes a to the singleton $\langle a \rangle$ and $\mu^{\mathcal{M}} : \mathcal{M}^2A \to \mathcal{M}A$ acts on objects by concatenation of sequences.

Each $\mathcal{M}A$ has the structure of a category with strict products: the terminal object is the empty sequence and product is given by concatenation. Again, one can check directly that $A \xrightarrow{\eta^{\mathcal{M}}} \mathcal{M}A$ makes $\mathcal{M}A$ the free category with strict products on A. Again, to equip A with the structure of a category with strict products is to give A a $\mathcal{M}$-algebra structure. Maps and 2-cells are as expected so we identify $\mathcal{M}\text{-}\mathbf{Alg}_s$ as the 2-category of categories with strict products, functors preserving these strictly and appropriate 2-cells.

3.3 The 2-monad for linear-non-linear substitution

There is a map $\lambda : \mathcal{L} \to \mathcal{M}$ which on objects takes $\langle a_i \rangle_{i\in[n]} \in \mathcal{L}A$ to $\langle a_i \rangle_{i\in[n]} \in \mathcal{M}A$ and includes the maps in $\mathcal{L}A$ into those in $\mathcal{M}A$ in the obvious way. It accounts for the evident fact that every category with strict product is a symmetric strict monoidal category. We describe the 2-monad $\mathcal{Q}$ obtained from this by λ by our colimit construction.

For a category A, $\mathcal{Q}A$ is the following category. The objects are $\langle a_i^{\epsilon_i} \rangle_{i\in[n]}$ with $n \in [n]$, $a_i \in A$ and the indices ϵ_i chosen from the set $\{\mathcal{L}, \mathcal{M}\}$ ($\mathcal{L}$ indicates linear and $\mathcal{M}$ non-linear). For $a = \langle a_i^{\epsilon_i} \rangle_{i\in[n]}$, write $\mathcal{L}_a$ for $\{i \,|\, \epsilon_i = \mathcal{L}\}$. Then a morphism

$$\langle a_i \rangle_{i\in[n]} \to \langle a'_j \rangle_{j\in[m]}$$

is given by first a map $\phi : [m] \to [n]$ satisfying the condition

$$\phi^{-1}(\mathcal{L}_a) \subseteq \mathcal{L}_b \quad \text{and} \quad \phi_{|\phi^{-1}(\mathcal{L}_a)} : \phi^{-1}(\mathcal{L}_a) \to \mathcal{L}_a \text{ is a bijection;}$$

and secondly by for each $j \in [m]$, a map $a_{\phi(j)} \to b_j$ in A.

$\mathcal{Q}$ extends readily to a 2-functor on **CAT** and it has the structure of a 2-monad as follows. The unit $\eta^{\mathcal{Q}} : A \to \mathcal{Q}A$ takes $a \in A$ to $\langle a^{\mathcal{L}} \rangle$. The multiplication $\mu^{\mathcal{Q}} : A \to \mathcal{Q}A$ acts by concatenating the objects and with the following behaviour on indices: objects of $\mathcal{Q}^2A$ have shape

$$\langle \langle \ldots \rangle \rangle \ldots \langle \ldots a^{\epsilon} \ldots \rangle^{\eta} \ldots \langle \ldots \rangle \rangle$$

so that each $a \in A$ has two indices; in the concatenated string in $\mathcal{Q}A$ a has index $\mathcal{L}$ just when both ϵ and η are $\mathcal{L}$.

One can now readily see the structure on $\mathcal{Q}A$ involved in its definition.

- $\mathcal{Q}A$ is clearly an $\mathcal{L}$-algebra and $k : \mathcal{L}A \to \mathcal{Q}A$ sends $\langle a_1, \ldots, a_n \rangle$ to $\langle a_1^{\mathcal{L}}, \ldots, a_n^{\mathcal{L}} \rangle$
- $\ell : \mathcal{M}A \to \mathcal{Q}A$ sends $\langle a_1, \ldots, a_n \rangle$ to $\langle a_1^{\mathcal{M}}, \ldots, a_n^{\mathcal{M}} \rangle$ given by the identity on $[n]$ and is evidently an $\mathcal{L}$-algebra map
- $\alpha : \ell\lambda \to k$ is given for each $\langle a_1, \ldots, a_n \rangle \in \mathcal{L}A$ by the map $\langle a_1^{\mathcal{M}}, \ldots, a_n^{\mathcal{M}} \rangle \to \langle a_1^{\mathcal{L}}, \ldots, a_n^{\mathcal{L}} \rangle$ given by the identity on $[n]$ and identities $a_i \to a_i$ for each i.

It is also easy to see $h : \mathcal{Q}A \to \mathcal{M}A$: it sends $\langle a_1^{\epsilon_1}, \ldots, a_n^{\epsilon_n} \rangle$ to $\langle a_1, \ldots, a_n \rangle$. It should now be straightforward for the reader to identify the 2-cell β and deduce that $\mu^{\mathcal{Q}}$ is just as described. Finally it is worth mulling over the content of our main theorem in this case.

4 Conclusion

Starting from the observation that the 2-monad $\mathcal{L}$ for strict monoidal categories and the 2-monad $\mathcal{M}$ for categories with strict products can be combined into a 2-monad $\mathcal{Q}$ mixing the two related structures, we have introduce a new notion for combining 2-monads as the colimit of a map of monads. We have proved that our construction gives rise to a 2-monad in Theorem 22 and characterised its algebras in Theorem 25.

Our next step will be to give conditions under which $\mathcal{Q}$ admits an extension to a pseudomonad on **Prof** [12]. That will give a basis for describing the substitution monoidal structure at play in differential λ-calculus [13].

We draw attention to the following issue which we need to address. It is clear from [12] that the 2-monad $\mathcal{L}$ for symmetric strict monoidal categories and $\mathcal{M}$ for categories with strict products admit extensions to pseudomonads on **Prof**. However, we cannot use our colimit construction at this level as we only have access to bicolimits. All the same, the characterisation of Theorem 25 will be useful to describe pseudo $\mathcal{Q}$-algebras. Then one can show that the presheaf construction has a lifting to pseudo $\mathcal{Q}$-algebras and so deduce by [12] the wanted extension of $\mathcal{Q}$ to **Prof**.

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