CSL

*Birmingham*

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# A linear logic approach to the semantics of probabilistic programs

joint work with **T. Ehrhard** and **M. Pagani**

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Study the *implementation* of probabilistic algorithms with *formal methods*: correctness, termination, contextual behaviour,...

**Bibliography**


| | | |
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| 2008 | Danos et al. | - Probabilistic coherent spaces |
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Differences: CBV or CBN evaluation, Discrete or Continuous data, first or higher order programs.

# Semantics of Probabilistic Programs

**Operational Semantics: how probabilistic programs compute**

The evaluation of a program is a markov process described by the probability of reduction from $M$ to $N$: $\mathbf{Prob}(M, N)$

- *Discrete type:* stochastic matrix
- *Continuous type:* stochastic kernel

**Denotational Semantics: invariant of computation**

If $M$ is a closed program, $[\![M]\!]$ can represent the results.

- *Discrete type ($\mathbb{N}$):* discrete distributions over integers
- *Continuous type ($\mathbb{R}$):* continuous distributions over reals

We will prove that the correctness of the implementation of two classic probabilistic algorithms in probability.

**Conditioning - handling discrete integers**

Given an array containing 0/1 cells, find the index of a 0 cell.

1. choose an index k
2. test if the content of the kth cell is 0
3. if yes output k
4. if no start from 1

Prove that `LV` outputs a correct value with probability 1

# Two examples of Probabilistic Programs

We will prove that the correctness of the implementation of two classic probabilistic algorithms in probability.

**Metropolis Hasting - handling continous reals**

Simulate a markov chain following a probabilistic law that we know only up to a scaling.

1. Start from a well-chosen point
2. Sample the proposal next point from a gaussian
3. Test if it is coherent with the previous one according to the wanted law up to a scaling
4. if yes use the proposal next point and start from 2
5. if no keep the previous point and start from 2

Prove that `MH` produces a markov chain following the wanted probabilistic law.

# What tools to study this programs

| | |
|---|---|
| Syntax | Describe programs, types and implementation |
| Operational semantics | Describe the evaluation of programs using $\mathbf{Prob}(M,N)$ a stochastic matrix or kernel |
| Denotational semantics | Interpret types using mathematical spaces<br>Interpret programs using mathematical functions |
| Invariance of semantics | Discrete: $[\![M]\!] = \sum_N \mathbf{Prob}(M,N)[\![N]\!]$<br>Continuous: $[\![M]\!] = \int \mathbf{Prob}(M,dt)[\![t]\!]$ |
| Adequacy Lemma | If $\vdash M : \mathtt{nat}$, then $[\![M]\!]_n = \mathbf{Prob}(M, \underline{n})$<br>If $\vdash M : \mathtt{real}$, then $[\![M]\!](U) = \mathbf{Prob}(M, \underline{U})$ |
| Adequacy | If $[\![P]\!] = [\![Q]\!]$ then $P \simeq Q$ (Discr.✓ / Cont.✓ ) |
| Full Abstraction | $[\![P]\!] = [\![Q]\!]$ iff $P \simeq Q$ (Discr.✓ / Cont.? ) |

1. **Discrete Probability**
   - Syntax: **Discrete** Probabilistic PCF
   - Semantics: **Pcoh** (Probabilistic Coherent Spaces)
   - Results: Probabilistic **Adequacy** & **Full Abstraction**
2. **Continuous Probability**

## Syntax of PPCF:

**Types:** $A, B ::= \mathtt{nat} \mid \mathtt{A} \to \mathtt{B}$

**Terms:** $M, N, L ::= x \mid \lambda x^A.M \mid (M)N \mid \mathbf{fix}(M) \mid$
$\mid \underline{n} \mid \mathsf{succ}(M) \mid \mathtt{ifz}(L, M, N) \mid \mathtt{let}\, x{=}M \,\mathtt{in}\, N$
$\mid \mathsf{coin}$

## Operational Semantics as a stochastic process: $M \xrightarrow{p} N$

$$
\begin{array}{rcl}
(\lambda x^A.M)N & \xrightarrow{1} & M[N/x] \\
\mathtt{ifz}(\underline{0}, M, N) & \xrightarrow{1} & M \\
\mathtt{ifz}(\underline{n+1}, M, N) & \xrightarrow{1} & N \\
\mathtt{let}\, x{=}\underline{n} \,\mathtt{in}\, N & \xrightarrow{1} & N[\underline{n}/x]
\end{array}
$$

# Syntax of $\mathrm{PPCF}$

## Syntax of $\mathrm{PPCF}$:

**Types:** $A, B ::= \mathtt{nat} \mid \mathtt{A} \to \mathtt{B}$

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$\mid \underline{n} \mid \mathsf{succ}(M) \mid \mathtt{ifz}(L, M, N) \mid \mathtt{let}\, x{=}M \,\mathtt{in}\, N$
$\mid \mathsf{coin}$

## Operational Semantics as a stochastic process: $M \xrightarrow{p} N$

$$\mathsf{coin} \xrightarrow{\frac{1}{2}} \underline{0} \qquad \mathsf{coin} \xrightarrow{\frac{1}{2}} \underline{1}$$

If $M \xrightarrow{p} M'$ then

$$\begin{aligned}
(M)N &\xrightarrow{p} (M')N \\
\mathtt{let}\, x{=}M \,\mathtt{in}\, N &\xrightarrow{p} \mathtt{let}\, x{=}M' \,\mathtt{in}\, N \\
\mathsf{succ}(M) &\xrightarrow{p} \mathsf{succ}(M') \\
\mathtt{ifz}(M, L, N) &\xrightarrow{p} \mathtt{ifz}(M', L, N), \ldots
\end{aligned}$$

## Syntax of PPCF:

**Types:** $A, B ::= \mathtt{nat} \mid \mathtt{A} \to \mathtt{B}$

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$\mid \mathsf{coin}$

## Operational Semantics as a stochastic matrix Prob$(\cdot,\cdot)$

$$\mathbf{Prob}((\lambda x^A.M)N, M[N/x]) = 1 : (\lambda x^A.M)N \xrightarrow{1} M[N/x]$$

$$\mathbf{Prob}(\mathsf{coin}, \underline{0}) = \mathbf{Prob}(\mathsf{coin}, \underline{1}) = \tfrac{1}{2} : \quad \mathsf{coin} \xrightarrow{\frac{1}{2}} \underline{0}, \quad \mathsf{coin} \xrightarrow{\frac{1}{2}} \underline{1}$$

### Syntax of PPCF:

**Types:** $A, B ::= \texttt{nat} \mid \texttt{A} \to \texttt{B}$

**Terms:** $M, N, L ::= x \mid \lambda x^A.M \mid (M)N \mid \mathbf{fix}(M) \mid$
$\mid \underline{n} \mid \mathsf{succ}(M) \mid \texttt{ifz}(L, M, N) \mid \texttt{let}\, x{=}M \,\texttt{in}\, N$
$\mid \mathsf{coin}$

### Operational Semantics as a stochastic matrix $\mathbf{Prob}(\cdot,\cdot)$

$\mathbf{Prob}(M, N)$: **probability** that $M \to N$ in **one** step.

$\mathbf{Prob}^2(M, N)$: **probability** that $M \to N$ in **two** steps.

...

$\mathbf{Prob}^\infty(M, N)$: **probability** that $M \to N$ in **any** steps
(when $N$ is a normal form)

### Syntax of PPCF:

**Types:** $A, B ::= \mathtt{nat} \mid \mathtt{A} \rightarrow \mathtt{B}$

**Terms:** $M, N, L ::= x \mid \lambda x^A.M \mid (M)N \mid \mathbf{fix}(M) \mid$
$\mid \underline{n} \mid \mathsf{succ}(M) \mid \mathtt{ifz}(L, M, N) \mid \mathtt{let}\, x{=}M \,\mathtt{in}\, N$
$\mid \mathsf{coin}$

### Operational Semantics as a stochastic matrix $\mathbf{Prob}(\cdot,\cdot)$

$$\mathbf{Prob}^2(M, N) = \sum_L \mathbf{Prob}(M, L)\mathbf{Prob}(L, N)$$

If $\vdash M : \mathtt{nat}$, then $\mathbf{Prob}^\infty(M, \_)$ is the subprobability **discrete distribution** over $\mathbb{N}$ of normal forms of $M$.

# How to encode a LasVegas Algorithm?

**Input:** A $\underline{0}/\underline{1}$ array of length $n \geq 2$ s.t. $\frac{1}{2}$ cells are $\underline{0}$.

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\underline{0}$ | $\underline{1}$ | $\underline{0}$ | $\underline{1}$ | $\underline{1}$ | $\underline{0}$ |

$$f: \begin{array}{lcl} 0,2,5 & \mapsto & \underline{0} \\ 1,3,4 & \mapsto & \underline{1} \end{array}$$

**Output:** Find the index of a cell containing $\underline{0}$.

**Caml:**

```
let rec LasVegas = let k = random n in
    if (f k = 0) then k
                 else LasVegas
```

**pPCF:**

$$\textbf{fix}\,(\lambda \texttt{LasVegas}^{\texttt{nat}}\ (\lambda \texttt{k}^{\texttt{nat}}$$
$$\texttt{ifz f k then k}$$
$$\texttt{else LasVegas})\ (\texttt{rand n})\,)$$

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|---|---|---|---|---|---|
| $\underline{0}$ | $\underline{1}$ | $\underline{0}$ | $\underline{1}$ | $\underline{1}$ | $\underline{0}$ |

$$f: \begin{array}{ccc} 0,2,5 & \mapsto & \underline{0} \\ 1,3,4 & \mapsto & \underline{1} \end{array}$$

**Output:** Find the index of a cell containing $\underline{0}$.

**Caml:**
**CBV**

```
let rec LasVegas  = let k = random n in
    if (f k = 0) then k
                 else LasVegas
```

**pPCF:**
**pure**
**CBN**

```
fix (λLasVegas^nat (λk^nat
        ifz f k then k
                 else LasVegas) (rand n) )
```

# How to encode a LasVegas Algorithm?

**Input:** A $\underline{0}/\underline{1}$ array of length $n \geq 2$ s.t. $\frac{1}{2}$ cells are $\underline{0}$.

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\underline{0}$ | $\underline{1}$ | $\underline{0}$ | $\underline{1}$ | $\underline{1}$ | $\underline{0}$ |

$$f: \begin{array}{lcl} 0,2,5 & \mapsto & \underline{0} \\ 1,3,4 & \mapsto & \underline{1} \end{array}$$

**Output:** Find the index of a cell containing $\underline{0}$.

**Caml:**
**CBV**

```
let rec LasVegas  = let k = random n in
    if (f k = 0) then k
                 else LasVegas
```

**pPCF:**
**let**

```
fix (λLasVegas^nat. let k = rand n in
        ifz (f k) then k
                  else LasVegas)
```

$$\begin{aligned} \texttt{LV} \;=\; \mathbf{fix}\,(\lambda \texttt{LasVegas}^{\texttt{nat}}\,.\ \texttt{let k = rand n in} \\ \texttt{ifz (f k) then k else LasVegas}) \end{aligned}$$

What is the probability `LV` terminates with a success: $\underline{k}$ such that $f(k) = 0$:

$$\begin{tikzcd} \texttt{LV} \arrow[r, "\frac{1}{2}"] \arrow[dr, "\frac{1}{2}"'] & \texttt{Success} & \\ & \texttt{LV} \arrow[r, "\frac{1}{2}"] \arrow[dr, "\frac{1}{2}"'] & \texttt{Success} \\ & & \texttt{LV}\ldots \end{tikzcd}$$

$$\mathbf{Prob}^{\infty}(\texttt{LV}, \texttt{Success}) = \sum_{k=1}^{\infty} \frac{1}{2^n} = 1$$

1. **Discrete Probability**
   - Syntax: **Discrete** Probabilistic PCF
   - Semantics: **Pcoh** (Probabilistic Coherent Spaces)
   - Results: Probabilistic **Adequacy** & **Full Abstraction**

2. **Continuous Probability**

| General Framework | Domains Semantics | Quantitative Semantics |
|---|---|---|
| **Types** | Continuous **dcpos** $(X, \leq)$ | **Proba.** spaces $(\lvert X\rvert, \mathrm{P}(X) \subseteq (\mathbb{R}^+)^{\lvert X\rvert})$ |
| **Programs** | **Scott Continuous** | **Analytic** Functions |
| **Probability** | Proba. **monad** | **Values** as proba. distr. |

**Bibliography**

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**Bibliography**

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| General Framework | Domains Semantics | Quantitative Semantics |
|---|---|---|
| Types | Continuous **dcpos** $(X, \leq)$ | **Proba.** spaces $(\lvert X\rvert, \mathrm{P}(X) \subseteq (\mathbb{R}^+)^{\lvert X\rvert})$ |
| Programs | **Scott Continuous** | **Analytic** Functions |
| Probability | Proba. **monad** | **Values** as proba. distr. |

**How to interpret a program** $M : \mathtt{nat} \Rightarrow \mathtt{nat}$

**Type:**
$\mathbb{N}_\perp$ flat domain,
$\mathcal{V}(\mathbb{N}_\perp)$ proba. distr. over $\mathbb{N}_\perp$,

**Prog:** $[\![M]\!] : \mathbb{N}_\perp \to \mathcal{V}(\mathbb{N}_\perp)$,
$[\![\mathtt{let\ n=x\ in\ M}]\!] : \mathcal{V}(\mathbb{N}_\perp) \to \mathcal{V}(\mathbb{N}_\perp)$

$$x \mapsto \left(\sum_n [\![M]\!]_{n,q} x_n\right)_q$$

**Type:**
$\lvert\mathbf{Nat}\rvert = \mathbb{N}$
$\mathrm{P}(\mathbf{Nat})$ subproba. dist. over $\mathbb{N}$

**Prog:** $[\![M]\!] : \mathrm{P}(\mathbf{Nat}) \to \mathrm{P}(\mathbf{Nat})$

$$x \mapsto \left(\sum_{\mu=[n_1,\ldots,n_k]} [\![M]\!]_{\mu,q} \prod_{i=1}^{k} x_{n_i}\right)_q$$

| General Framework | Domains Semantics | Quantitative Semantics |
|---|---|---|
| **Types** | Continuous **dcpos** $(X, \leq)$ | **Proba.** spaces $(\lvert X\rvert, \mathrm{P}(X) \subseteq (\mathbb{R}^+)^{\lvert X\rvert})$ |
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### Problematic in **domain**

Finding a full subcategory of continuous dcpos that is: **Cartesian Closed** and **closed** under the proba. monad $\mathcal{V}$.

### Full Abs.: **PCOH/pPCF**

$$\mathbf{Prob}(C[M], \underline{n}) \underset{=}{\forall n, \forall C[]} \mathbf{Prob}(C[N], \underline{n})$$

iff

$$[\![M]\!] = [\![N]\!].$$

# Types as **Probabilistic Coherent Spaces**: $(|X|, P(X))$

## Proba. Space

$|X|$: the **web**, a (potentially infinite) set of final states

$\mathrm{P}(X)$: a set of vectors $\subseteq (\mathbb{R}^+)^{|X|}$ such that

**closure:** $\mathbf{P}(\mathbf{X})^{\perp\perp} = \mathbf{P}(\mathbf{X})$ with
$\forall u, v \in (\mathbb{R}^+)^{|X|},\ \langle u, v\rangle = \sum_{a\in|X|} u_a v_a$
$\forall P \subseteq (\mathbb{R}^+)^{|X|},\ P^\perp = \{v \in (\mathbb{R}^+)^{|X|}\ ;\ \forall u \in P,\ \langle u, v\rangle \leq 1\}$

**bounded covering:** $\forall a \in |X|$,
$\exists v \in P(X)\ ;\ v_a \neq 0$ and $\exists p > 0,\ ;\ \forall v \in \mathrm{P}(X),\ v_a \leq p.$

## Proposition: Proba. spaces as Domains

$(|X|, \mathrm{P}(X))$ is a **Proba. space iff** $\mathrm{P}(X)$ is bounded covering, **Scott Closed** (downwards-closed and dcpo) and **Convex**.

# Types as **Probabilistic Coherent Spaces**: $(|X|, P(X))$

**Example:** $P(X) \subseteq (\mathbb{R}^+)^{|X|}$

$$|\mathbf{1}| = \{*\} \qquad \mathrm{P}(\mathbf{1}) = [0,1]$$

$$|\mathbf{Bool}| = \{t, f\} \qquad \mathrm{P}(\mathbf{Bool}) = \{(x_t, x_f)\ ;\ x_t + x_f \leq 1\}$$

$$|\mathbf{Nat}| = \{0, 1, 2, \dots\} \qquad \mathrm{P}(\mathbf{Nat}) = \{x \in [0,1]^{\mathbb{N}}\ ;\ \sum_n x_n \leq 1\}$$

$$|\mathbf{Bool} \to \mathbf{1}| = \{[t^n, f^m]\ ;\ n, m \in \mathbb{N}\},$$

$$\mathrm{P}(\mathbf{Bool} \to \mathbf{1}) = \{Q \in (\mathbb{R}^+)^{|\mathbf{Bool} \to \mathbf{1}|}\ ;$$

$$\forall x_t + x_f \leq 1,\quad \sum_{n,m=0}^{\infty} Q_{[t^n, f^m]}\, x_t^n x_f^m \leq 1\}$$

**Proposition: Proba. spaces as Domains**

$(|X|, \mathrm{P}(X))$ is a **Proba. space iff** $\mathrm{P}(X)$ is bounded covering, **Scott Closed** (downwards-closed and dcpo) and **Convex**.

**Pcoh : Linear Category**

Objects: Proba. Spaces

Morphisms: Linear Functions

**Call by Name** $A \to B = !A \multimap B$

**$\mathbf{Pcoh}_!$: Kleisli Category**

Objects: Proba. Spaces

Morphisms: Analytic Functions

- **Smcc** $(\mathbf{1}, \otimes, \multimap)$
- biproduct

- **Comonad** $(\mathbf{!}, \mathrm{der}, \mathrm{dig})$
- **Com. Comonoid** $(!A, \mathbf{1}, \otimes)$

- **CCC**
- (PCF+coin)

**Pcoh**$(X, Y)$

Matrices $Q \in (\mathbb{R}^+)^{|X|\times|Y|}$ such that:

$$\forall x \in \mathrm{P}(X),\ Q \cdot x = \left( \sum_{a \in |X|} Q_{a,b}\, x_a \right)_b \in \mathrm{P}(Y)$$

Example

**Pcoh(Nat, Nat)**: Stochastic Matrices $Q \in (\mathbb{R}^+)^{\mathbb{N}\times\mathbb{N}}$.

$$\forall x \in (\mathbb{R}^+)^{\mathbb{N}}\ ;\ \sum_{n\in\mathbb{N}} x_n \leq 1,\ \sum_{m,n\in\mathbb{N}} Q_{m,n} x_n \leq 1$$

# Free Commutative Comonoid and Comonad

**Exponential**

$$|!X| = \mathcal{M}_{\mathsf{fin}}(|X|) \text{ the set of finite multisets}$$

$$\mathrm{P}(!X) = \{x^! \;;\; x \in \mathrm{P}(X)\}^{\perp\perp} \text{ where } x^!_{[a_1,\dots,a_k]} = \prod_{i=1}^k x_{a_i}$$

**Example**

Let $\mathbf{Bcoin} = (p, 1-p) \in \mathrm{P}(\mathbf{Bool}) = \{(p,q) \;;\; p+q \leq 1\}$.

$$\mathbf{Bcoin}^!_{[]} = 1, \qquad \mathbf{Bcoin}^!_{[t,t]} = p^2, \qquad \mathbf{Bcoin}^!_{[t,f]} = p(1-p), \;\dots$$

**Theorem (2017: Crubillé - Ehrhard - Pagani - T.)**

This exponential computes the free commutative comonoid.

# Free Commutative Comonoid and Comonad

**Exponential**

$$|!X| = \mathcal{M}_{\mathrm{fin}}(|X|) \text{ the set of finite multisets}$$

$$\mathrm{P}(!X) = \{x^! \ ; \ x \in \mathrm{P}(X)\}^{\perp\perp} \text{ where } x^!_{[a_1,\dots,a_k]} = \prod_{i=1}^k x_{a_i}$$

**Commutative Comonoid** | **Comonad**

**Cocontr.**: $!X \xrightarrow{c^{!X}} !X \otimes !X$

**Coweak.**: $!X \xrightarrow{w^{!X}} \mathbf{1}$

**Comult.:** $\mathrm{dig}_{!X} : !!X \to !X$

**Counit:** $\mathrm{der}_{!X} : !X \to X$

**Theorem (2017: Crubillé - Ehrhard - Pagani - T.)**

This exponential computes the free commutative comonoid.

**$\mathbf{Pcoh}_!(X, Y) = \mathbf{Pcoh}(!X, Y)$**

Matrices $Q \in (\mathbb{R}^+)^{\mathcal{M}_{\mathrm{fin}}(|X|)\times|Y|}$ such that

$$\forall U \in \mathrm{P}\,(!X),\ \ Q\cdot U = \left(\sum_{m\in\mathcal{M}_{\mathrm{fin}}(|X|)} Q_{m,b}\, U_m\right)_b \in \mathrm{P}\,(Y)$$

Non-Linear Morphisms are **analytic** and **Scott Continuous**.

$$\mathbf{Pcoh}_!(\mathbf{Bool}, \mathbf{1}) = \{Q \in (\mathbb{R}^+)^{|\mathbf{Bool}\to\mathbf{1}|}\ \ s.t.\ \ Q_{[t^n, f^m]} \leq \frac{(n+m)^{n+m}}{n^n\, m^m}\}$$

```
let rec f x =
 if x then if x then f x
                else ()
      else if x then ()
                else f x
```

denotes

$$\sum_{n,m=0}^{\infty} \frac{(n+m)!}{n!\, m!} x_t^{2n+1} x_f^{2m+1}$$

**$\mathbf{Pcoh}_!(X,Y) = \mathbf{Pcoh}(!X,Y)$** **Density**

Matrices $Q \in (\mathbb{R}^+)^{\mathcal{M}_{\text{fin}}(|X|)\times|Y|}$ such that if $x^!_m = \prod_{a\in m} x_a^{m(a)}$

$$\forall x \in \mathrm{P}(X),\ \widehat{Q}(x) = \left(\sum_{m\in\mathcal{M}_{\text{fin}}(|X|)} Q_{m,b}\, x^!{}_m\right)_b \in \mathrm{P}(Y)$$

Non-Linear Morphisms are **analytic** and **Scott Continuous**.

$\mathbf{Pcoh}_!(\mathbf{Bool},\mathbf{1}) = \{Q \in (\mathbb{R}^+)^{|\mathbf{Bool}\to\mathbf{1}|}\ s.t.\ Q_{[t^n,f^m]} \leq \frac{(n+m)^{n+m}}{n^n\, m^m}\}$

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**Pcoh**$_!(X,Y) = $ **Pcoh**$(!X,Y)$ **Density**

Matrices $Q \in (\mathbb{R}^+)^{\mathcal{M}_{\mathrm{fin}}(|X|)\times|Y|}$ such that if $x^!_m = \prod_{a\in m} x_a^{m(a)}$

$$\forall x \in \mathrm{P}(X),\ \widehat{Q}(x) = \left(\sum_{m\in\mathcal{M}_{\mathrm{fin}}(|X|)} Q_{m,b}\, x^!{}_m\right)_b \in \mathrm{P}(Y)$$

Non-Linear Morphisms are **analytic** and **Scott Continuous**.

**Pcoh**$_!$(**Bool**, **1**) $= \{Q \in (\mathbb{R}^+)^{|\mathbf{Bool}\to\mathbf{1}|}\ s.t.\ Q_{[t^n,f^m]} \le \frac{(n+m)^{n+m}}{n^n\, m^m}\}$

```
let rec f x =
 if x then if x then f x
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      else if x then ()
                 else f x
```

**pb of DEFINABILITY**

$$\sum_{n,m=0}^{\infty} \frac{(n+m)!}{n!\,m!} x_t^{2n+1} x_f^{2m+1}$$

If $\Gamma \vdash M : A$, then $[\![A]\!]^{\Gamma} \in \mathbf{Pcoh}_!(\Gamma, A)$

$\vdash \underline{n} : \mathtt{nat}$, thus $[\![n]\!] \in \mathrm{P}\left(\mathbf{Nat}\right)$ is a distribution over $\mathbb{N}$:

$$[\![\underline{n}]\!] = (0, \ldots, 0, \; 1 \; , 0, \ldots) \qquad n\text{th}$$

$\vdash \mathtt{rand\ n} : \mathtt{nat}$, thus $[\![\mathtt{rand\ n}]\!]$ is a distribution over $\mathbb{N}$:

$$[\![\mathtt{rand\ n}]\!] = (\tfrac{1}{n}, \ldots, \; \tfrac{1}{n} \; , 0, \ldots) \qquad (n-1)\text{th}$$

If $\vdash N : \mathtt{nat}$ and $\vdash P : A$ and $\vdash Q : A$, then

$$[\![\mathtt{ifz}(N, P, Q)]\!] \;=\; [\![N]\!]_0 [\![P]\!] + \sum_{k=0}^{\infty} [\![N]\!]_{k+1} [\![Q]\!]$$

$$[\![\mathtt{let}\, x{=}N\, \mathtt{in}\, P]\!] \;=\; \sum_{k=0}^{\infty} [\![N]\!]_k \widehat{[\![P]\!]}(k)$$

1. **Discrete Probability**
   - Syntax: **Discrete** Probabilistic PCF
   - Semantics: **Pcoh** (Probabilistic Coherent Spaces)
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2. **Continuous Probability**

# **First results** [Danos-Ehrhard 2011]

| | |
|---|---|
| Syntax | pPCF |
| Operational semantics | $\mathbf{Prob}(M,N) = p$ iff $M \overset{p}{\to} N$<br>stochastic matrix vs. stochastic process |
| Denotational semantics | **Types** as probabilistic spaces: $[\![A]\!] = (\lvert A\rvert, \mathrm{P}(A))$<br>**Programs** as **analytic functions**:<br>if $A \vdash M : B$ then $\widehat{[\![M]\!]} : \mathrm{P}(A) \to \mathrm{P}(B)$ |

$$\forall x \in \mathrm{P}(A), \forall b \in \lvert B\rvert, \widehat{[\![M]\!]}(x)_b = \sum_{m \in \mathcal{M}_{\mathrm{fin}}(\lvert A\rvert)} [\![M]\!]_{m,b} \prod_{a \in m} x_a^{m(a)}$$

| | |
|---|---|
| Compositionality | $[\![(M)N]\!]_b = \widehat{[\![M]\!]}([\![N]\!])_b = \sum_m [\![M]\!]_{m,b} \prod_{a \in m} [\![N]\!]_a^{m(a)}$ |
| Invariance of sem. | $[\![M]\!] = \sum_N \mathbf{Prob}(M,N)[\![N]\!]$ |
| Adequacy Lemma | if $\vdash M : \mathtt{nat}$, then $\mathbf{Prob}^{\infty}(M,\underline{n}) = [\![M]\!]_n$ |

Theorem (2014: Ehrhard - Pagani - T.)

| **Pcoh** | | **pPCF** |
|---|---|---|
| $[\![M]\!] = [\![N]\!]$ | Adequacy $\Longrightarrow$ <br> $\Longleftarrow$ Full Abstraction | $M \simeq_o N$ <br> $\mathbf{Prob}^\infty(C[M], n) \overset{\forall C[\,]\forall n}{=} \mathbf{Prob}^\infty(C[N], n)$ |

**Adequacy proof:**

If $[\![M]\!] = [\![N]\!]$ then, $\mathbf{Prob}^\infty((C)M, \underline{n}) = \mathbf{Prob}^\infty((C)N, \underline{n})$

1. Apply Adequacy Lemma : $\mathbf{Prob}^\infty((C)M, \underline{n}) = [\![(C)M]\!]_n$.
2. Apply Compositionality:

$$[\![(C)M]\!]_n = \sum_m [\![C]\!]_{m,n} \prod_{a \in m} [\![M]\!]_a^{m(a)} = \sum_m [\![C]\!]_{m,n} \prod_{a \in m} [\![N]\!]_a^{m(a)} = [\![(C)N]\!]_n$$

**Theorem (2014: Ehrhard - Pagani - T.)**

| **Pcoh** | | **pPCF** |
|---|---|---|
| $[\![M]\!] = [\![N]\!]$ | Adequacy $\Longrightarrow$ $\Longleftarrow$ Full Abstraction | $M \simeq_o N$ |
| | | $\mathbf{Prob}^{\infty}(C[M], n) \overset{\forall C[\,]\forall n}{=} \mathbf{Prob}^{\infty}(C[N], n)$ |

## Full Abstraction Proof:

1. By **contradiction**: $\exists \alpha \in |\sigma|,\ [\![M]\!]_\alpha \neq [\![N]\!]_\alpha$
2. Find **testing context**: $T_\alpha$ such that $[\![(T_\alpha)M]\!] \neq [\![(T_\alpha)N]\!]$ (context only depends on $\alpha$)
3. Prove **definability**: $T_\alpha \in$ pPCF using coin and regularity of analytic functions
4. Apply **Adequacy Lemma**:
   $\mathbf{Prob}((T_\alpha)M \xrightarrow{*} \underline{0}) \neq \mathbf{Prob}((T_\alpha)N \xrightarrow{*} \underline{0})$.

$$\texttt{LV} \;=\; \textbf{fix}\,(\lambda\texttt{LasVegas}^{\texttt{nat}}.\ \texttt{let k = rand n in}$$
$$\texttt{ifz (f k) then k else LasVegas})$$

**Input:** A $\underline{0}/\underline{1}$ array of length $n \geq 2$ s.t. $\frac{1}{2}$ cells are $\underline{0}$.

| 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $\underline{0}$ | $\underline{1}$ | $\underline{0}$ | $\underline{1}$ | $\underline{1}$ | $\underline{0}$ |

$$f:\quad \begin{array}{lcl} 0,2,5 & \mapsto & \underline{0} \\ 1,3,4 & \mapsto & \underline{1} \end{array}$$

**Output:** Find the index of a cell containing $\underline{0}$.

We want to prove that $\mathbf{Prob}^{\infty}(\texttt{LV}, \texttt{Success}) = 1$

$$\mathtt{LV} = \textbf{fix}\,(\lambda\mathtt{LasVegas}^{\mathtt{nat}}\,.\ \mathtt{let\ k\ =\ rand\ n\ in}$$
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By operational semantics:

$$\mathtt{LV} \xrightarrow{1} \mathtt{let}\ \mathtt{k} = \mathtt{rand}\ \mathtt{n}\ \mathtt{in}\ \mathtt{ifz}\ (\mathtt{f}\,\mathtt{k})\ \mathtt{then}\ \underline{\mathtt{k}}\ \mathtt{else}\ \mathtt{LV}$$

# Semantical proof of correction of LasVegas

$$\mathtt{LV} \;=\; \mathbf{fix}\,(\lambda \mathtt{LasVegas}^{\mathtt{nat}}\,.\ \mathtt{let\ k = rand\ n\ in}\ \mathtt{ifz\ (f\ k)\ then\ k\ else\ LasVegas})$$

By operational semantics:

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By invariance of the semantics and interpretation of `let` and `ifz`:

$$\begin{aligned}
[\![\mathtt{LV}]\!]_p &= \sum_{k=0}^{\infty} [\![\mathtt{rand\,n}]\!]_k [\![\mathtt{ifz\,(f\,k)\,then\,\underline{k}\,else\,LV}]\!]_p \\
&= \frac{1}{n} \cdot \left( \sum_{f(k)=0 k<n} [\![\underline{k}]\!]_p + \sum_{f(k)\neq 0 k<n} [\![\mathtt{LV}]\!]_p \right)
\end{aligned}$$

If $p < n \;\&\; f(p) = 0$, then $[\![\mathtt{LV}]\!]_p = \frac{1}{n} + \frac{1}{n} \cdot \frac{n}{2} \cdot [\![\mathtt{LV}]\!]_p$, so $[\![\mathtt{LV}]\!]_p = \frac{2}{n}$.

If $p \geq n$ or $f(p) \neq 0$, then $[\![\mathtt{LV}]\!]_p = \frac{1}{n} \cdot \frac{n}{2} \cdot [\![\mathtt{LV}]\!]_p$, so $[\![\mathtt{LV}]\!]_p = 0$.

# Semantical proof of correction of LasVegas

$$\texttt{LV} \;=\; \textbf{fix}\,(\lambda \texttt{LasVegas}^{\texttt{nat}}.\ \texttt{let k = rand n in}$$
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If $p < n$ and $f(p) = 0$, then $[\![\texttt{LV}]\!]_p = \frac{2}{n}$, otherwise $[\![\texttt{LV}]\!]_p = 0$.

$$\mathtt{LV} \;=\; \mathbf{fix}\,(\lambda \mathtt{LasVegas}^{\mathtt{nat}}\,.\ \mathtt{let\ k\ =\ rand\ n\ in}$$
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If $p < n$ and $f(p) = 0$, then $[\![\mathtt{LV}]\!]_p = \frac{2}{n}$, otherwise $[\![\mathtt{LV}]\!]_p = 0$.

Using Adequacy Lemma, the probability that LV converges:

$$\begin{aligned}
\mathbf{Prob}^{\infty}(\mathtt{LV}, \mathtt{Success}) &= \sum_{p} \mathbf{Prob}^{\infty}(\mathtt{LV}, \underline{p}) \\
&= \sum_{p} [\![\mathtt{LV}]\!]_p \\
&= \sum_{\substack{f(p)=0 \\ p<n}} \frac{2}{n} = \frac{n}{2} \cdot \frac{2}{n} \\
&= 1
\end{aligned}$$

1. Discrete Probability

2. **Continuous Probability**
   - Syntax: **Real** Probabilistic PCF
   - Semantics: $\mathbf{Cstab}_{\mathrm{m}}$ (Cones and Stable measurable functions)
   - Results: **Adequacy**

## Nat PPCF

**Types:** $A, B ::= \texttt{nat} \mid A \to B$

**Terms:** $M, N, L ::=$
$x \mid \lambda x^A.M \mid (M)N \mid \mathbf{fix}(M) \mid$
$\underline{n} \mid \mathsf{succ}(M) \mid$
$\texttt{ifz}(L, M, N) \mid$
$\mathsf{coin} \mid \texttt{let}\, x{=}M\, \texttt{in}\, N$

**Operational Semantics:**
$\mathbf{Prob}(\mathsf{coin}, \underline{0}) = \frac{1}{2}$

If $\vdash M : \texttt{nat}$, $\mathbf{Prob}^\infty(M, \_)$ is the discrete distribution over $\mathbb{N}$ computed by $M$.

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## Real PPCF

**Types:** $A, B ::= \mathtt{real} \mid A \to B$

**Terms:** $M, N, L ::=$

$x \mid \lambda x^A.M \mid (M)N \mid \mathbf{fix}(M) \mid$

$\underline{r} \mid \underline{f}(M_1, \ldots, M_n) \mid$

$\mathtt{ifz}(L, M, N) \mid$

$\mathtt{sample} \mid \mathtt{let}\, x{=}M\, \mathtt{in}\, N$

**Operational Semantics:**

$\mathbf{Prob}(\mathtt{sample}, U) = \lambda_{[0,1]}(U)$

If $\vdash M : \mathtt{real}$, $\mathbf{Prob}^\infty(M, \_)$ is the continuous distribution over $\mathbb{R}$ computed by $M$.

The probability to observe $U$ after at most one reduction step applied to $M$ is **Prob**$( M , U )$

**Prob** $: \Lambda^{\Gamma\vdash A} \times \Sigma_{\Lambda^{\Gamma\vdash A}} \to \mathbb{R}^+$ is a stochastic **Kernel**, i.e:

- for all $M \in \Lambda^{\Gamma\vdash A}$, **Prob**$(M, \_)$ is a measure;
- for all $U \in \Sigma_{\Lambda^{\Gamma\vdash A}}$, **Prob**$(\_, U)$ is a measurable function.

**Prob**$^{\infty}(M, U)$ is the probability to observe $U$ after any steps.

The probability to observe $U$ after at most one reduction step applied to $M$ is **Prob**$(\ M\ ,\ U\ )$

$\Lambda^{\Gamma\vdash A}$: the set of terms $M$ s.t. $\Gamma \vdash M : A$.

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**Measurable sets and kernels constitute the category Kern.**

**Prob**$^\infty(M, U)$ is the probability to observe $U$ after any steps.

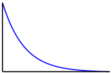
It is computed by composition and lub.

The Bernoulli distribution takes the value 1 with probability $p$ and the value 0 with probability $1 - p$.

$p\delta_1 + (1-p)\delta_0$

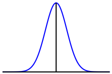
$\texttt{bernoulli}\ p ::= \texttt{let}\ x{=}\texttt{sample}\ \texttt{in}\ x \leq p$
tests if `sample` draws a value within $[0, p]$.

The exponential distribution is specified by its density $\mathrm{e}^{-x}$.



$\texttt{exp} ::= \texttt{let}\ x{=}\texttt{sample}\ \texttt{in} - \log(x)$
by the inversion sampling method.

The standard normal distribution defined by its density $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}$.



$\texttt{gauss} ::= \texttt{let}\ x{=}\texttt{sample}\ \texttt{in}$
$\texttt{let}\ y{=}\texttt{sample}\ \texttt{in}\ \sqrt{-2\log(x)}\ \cos(2\pi y)$
by the Box Muller method.

Conditioning: If $U \subseteq \mathbb{R}$ measurable, then $\mathtt{observe}(U)$ of type $\mathtt{real} \to \mathtt{real}$, taking a term $M$ and returning the renormalization of the distribution of $M$ on the only samples that satisfy $U$: conditioning by rejection sampling.

$$\mathtt{observe}(U) = \lambda m.\, \mathbf{fix}(\lambda y.\mathtt{let}\, x{=}m\, \mathtt{in}\, \mathtt{if}(x \in U, x, y))$$

Monte Carlo Simulation,...

# How to encode Metropolis Hasting

**Input:** $\mu$ a distribution on $\mathbb{R}$ with density $\pi$: $\mu(U) = \int_U \pi(x)dx$, but we only know $\gamma\pi$.

**Output:** Markov Chain $x_n$ converging to a random variable $x$ with law $\mu$

1. Initialized $x$ with a well-chosen point $\mathtt{x}_0$
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\quad\quad \texttt{let x = MetHast (n-1) in} \\
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\quad\quad\quad\quad\quad\quad\quad\quad \texttt{if } z = 0 \texttt{ then x else y})
\end{array}
$$

1. Discrete Probability

2. **Continuous Probability**
   - Syntax: **Real** Probabilistic PCF
   - Semantics: $\mathbf{Cstab_m}$ (Cones and Stable measurable functions)
   - Results: **Adequacy**

1981, Kozen Memory as measurable space and programs as kernels representing the transformation of the memory.
What is a measurable subset for function space ?

1999, Panangaden
**Meas**, the category of measurable sets and functions
**Kern**, the category of measurable sets and kernels
They are **cartesian** but **not closed**.

2017, Heunen, Kammar, Staton, Yang **Quasi-borel spaces**
A **CCC** based on **Meas** embedded into presheaves.
How to interpret recursive types ?

2017, Keimel and Plotkin **Kegelspitzen**
A **CCC** of dcpos equipped with a convex structure (basic operations being scott continous) with scott continuous functions
How to restrict to measurable functions ?

| **Discrete** | **Continuous** |
|---|---|
| If $\vdash M : \mathtt{nat}$, then $[\![M]\!]$ is a distribution over $\mathbb{N}$ | If $\vdash M : \mathtt{real}$, then $[\![M]\!]$ is a measure over $\mathbb{R}$ |

- $[\![\mathtt{real}]\!]$ as $\mathrm{Meas}(\mathbb{R})$ the set of measures over $\mathbb{R}$.
- Fixpoint of terms.

$\mathbf{Cstab_m}$ is a **CCC** based on Selinger's **cones** (dcpos with the order induced by addition and a convex structure).

**Objects** are cones and measurable spaces

**Morphisms** are stable and measurable functions

**Pcoh** is a subcategory of $\mathbf{Cstab_m}$ which is a subcategory of Kegelspitzen.

Our purpose is to be able to interpret `real` as the set of bounded measures.

1. **Complete cones** (convex dcpos with the order induced by addition) with Scott continuous functions
   However, the category is cartesian but not closed.
2. Complete cones and **Stable functions** ($\infty$-non-decreasing functions) is a CCC.
   However, not every stable function is measurable.
3. **Measurable Cones** (complete cones with **measurable tests**). Measurable paths pass measurable tests and Measurable functions preserve measurable paths.
   **$\mathbf{Cstab_m}$ is a CCC with measurability included !**

**$\mathbf{Pcoh}_!$**

- For $\vdash \underline{n} : \mathbb{N}$,
  $[\![\underline{n}]\!]_p = \delta_{p,n}$
- For $\vdash \mathrm{coin} : \mathbb{N}$,
  $[\![\mathrm{coin}]\!]_p = \frac{1}{2}\delta_{0,p} + \frac{1}{2}\delta_{1,p}$
- For $\vdash N : \mathbb{N}$, $\vdash P : A$, $\vdash Q : A$,
  $[\![\mathtt{ifz}(N, P, Q)]\!]_a =$
  $[\![N]\!]_0 [\![P]\!]_a + \sum_{n \neq 0} [\![N]\!]_{n+1} [\![Q]\!]_a$

  $[\![\mathtt{let}\, x{=}N\, \mathtt{in}\, P]\!]_a =$
  $$\sum_{n=0}^{\infty} [\![N]\!]_n \widehat{[\![P]\!]}(n)_a$$

**$\mathbf{Pcoh}_!$**

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$$[\![\mathtt{let}\, x{=}N \,\mathtt{in}\, P]\!]_a = \sum_{n=0}^{\infty}[\![N]\!]_n\widehat{[\![P]\!]}(n)_a$$

**$\mathbf{Cstab}_\mathrm{m}$**

- For $\vdash \underline{r} : \mathtt{real}$,
  $[\![\underline{r}]\!](U) = \delta_r(U)$
- For $\vdash \mathtt{sample} : \mathtt{real}$,
  $[\![\mathtt{sample}]\!] = \lambda_{[0,1]}(U)$
- For $\vdash R : \mathtt{real}$, $\vdash P, Q : A$,
  $[\![\mathtt{ifz}(R,P,Q)]\!](U) =$
  $[\![R]\!](\{0\})[\![P]\!](U) + [\![R]\!](\mathbb{R}\setminus\{0\})[\![Q]\!](U)$

$$[\![\mathtt{let}\, x{=}R \,\mathtt{in}\, P]\!](U) = \int [\![R]\!](dr)[\![P]\!](\delta_r)(U)$$

1 Discrete Probability

2 **Continuous Probability**

- Syntax: **Real** Probabilistic PCF
- Semantics: $\mathbf{Cstab}_{\mathrm{m}}$ (Cones and Stable measurable functions)
- Results: **Adequacy**

The category $\mathbf{Cstab_m}$ is a CCC and a model of Real PPCF.

**Invariance of the semantics**

$$[\![M]\!]^{\Gamma\vdash A} = \int_{\Lambda^{\Gamma\vdash A}} [\![t]\!]^{\Gamma\vdash A}\mathbf{Prob}(M, dt)$$

**Adequacy**

$$[\![M]\!]^{\vdash \mathtt{real}}(U) = \mathbf{Prob}^{\infty}(M, U)$$

Full Abstraction ??

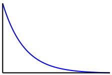
The Bernoulli distribution takes the value 1 with probability $p$ and the value 0 with probability $1-p$.

$$\texttt{bernoulli}\ p ::= \texttt{let}\ x{=}\texttt{sample in}\ x{\le}p$$

$p\delta_1{+}(1{-}p)\delta_0$

$$[\![\texttt{bernoulli}\ \underline{p}]\!]^{\vdash\texttt{real}} = p\delta_1 + (1-p)\delta_0$$

The exponential distribution is specified by its density $\mathrm{e}^{-x}$.



$$\mathsf{exp} : \texttt{real} ::= \texttt{let}\ x{=}\texttt{sample in} - \log(x)$$

$$[\![\mathsf{exp}]\!]^{\vdash\texttt{real}}(U) = \int_{\mathbb{R}^+} \chi_U(s)\mathrm{e}^{-s}\lambda(ds)$$

The standard normal distribution defined by its density $\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}x^2}$.

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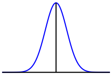

$$[\![\texttt{gauss}]\!]^{\vdash\texttt{real}}(U) = \frac{1}{\sqrt{2\pi}}\int_U \mathrm{e}^{-\frac{x^2}{2}}\lambda(dx)$$

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$\texttt{observe}(U) = \lambda m.\, \mathbf{fix}(()\lambda y.\texttt{let}\, x{=}m\, \texttt{in}\, \texttt{if}(x \in U, x, y))$ conditioning by rejection sampling.
Whenever $M$ represents a probability distribution, this equation gives the conditional probability:

$$[\![\texttt{observe}(U) M]\!](V) = \frac{[\![M]\!](V \cap U)}{[\![M]\!](U)}$$

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      let x = MetHast (n-1) in
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                  if z = 0 then x else y)
```

$\mathtt{MH}(\underline{0}) \to \mathtt{x}_0$ thus, $\mathbf{Prob}(\mathtt{MH}(\underline{0}), U) = \delta_{x_0}(U)$

$\mathtt{MH}(\underline{n+1}) \to M = \mathtt{let}\, x{=}\mathtt{MH}(\underline{n})\,\mathtt{in}\,\mathtt{let}\, y{=}\mathtt{gauss}\, x\,\mathtt{in}$
$\mathtt{let}\, z{=}\mathtt{bernoulli}(\underline{\alpha}(x,y))\,\mathtt{in}\,\mathtt{ifz}(z,x,y)$

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$$\texttt{MH}(\underline{n+1}) \to M = \texttt{let}\, x{=}\texttt{MH}(\underline{n})\,\texttt{in}\,\texttt{let}\, y{=}\texttt{gauss}\, x\,\texttt{in}$$
$$\texttt{let}\, z{=}\texttt{bernoulli}(\underline{\alpha}(x,y))\,\texttt{in}\,\texttt{ifz}(z,x,y)$$

$$\mathbf{Prob}(\texttt{MH}(\underline{n+1}), U) = [\![\texttt{MH}(\underline{n+1})]\!](U) = [\![M]\!](U) \text{ (Adequacy/Reduction)}$$
$$= \int_{\mathbb{R}} [\![N]\!](\delta_r)(U)\, [\![\texttt{MH}(\underline{n})]\!](dr) = \int_{\mathbb{R}} P_{\texttt{MH}}(r,U)\, \mathbf{Prob}(\texttt{MH}(\underline{n}), dr)$$

$$P_{\texttt{MH}}(r,U) = \delta_r(U)\left(1 - \int_{\mathbb{R}} \alpha(r,t) g(t,r) \lambda(dt)\right) + \int_U \alpha(r,t) g(t,r) \lambda(dt).$$

# How to encode Metropolis Hasting

**Input:** $\mu$ a distribution on $\mathbb{R}$ with density $\pi$:
$\mu(U) = \int_U \pi(x)dx$, but we only know $\gamma\pi$.

**Output:** Markov Chain $x_n$ converging to
a random variable $x$ with law $\mu$

$$\mathbf{Prob}(\mathtt{MH}(\underline{n+1}), U) = \int_{\mathbb{R}} P_{\mathtt{MH}}(r, U)\, \mathbf{Prob}(\mathtt{MH}(\underline{n}), dr),$$

$$P_{\mathtt{MH}}(r, U) = \delta_r(U) \left(1 - \int_{\mathbb{R}} \alpha(r,t)g(t,r)\lambda(dt)\right) + \int_U \alpha(r,t)g(t,r)\lambda(dt).$$

This shows that $\mathbf{x}_n$ is a Markov-Chain whose law is defined with respect to the kernel $P_{\mathtt{MH}}(r, U)$. It is standard mathematics to prove that $\mu$ is its invariant measure.

***A denotational semantics for probabilistic higher-order functional computation,***

(based on **quantitative** semantics of Linear Logic)

**Discrete setting:**

Probabilistic Coherent Spaces are **fully abstract** for a programming language with **natural numbers** as base types suitable to encode discrete probabilistic programs.

**Continuous setting:**

A **CCC** of measurable spaces and **stable** maps that soundly denotes a programming language with **reals** as base types suitable to encode continuous probabilistic programs.

**Storage Operator**

```
let k = rand n in if k = 0 then k else 42
```

**Integer in Pcoh:** $[\![\mathtt{nat}]\!] = \mathbf{Nat} = (\mathbb{N}, \mathrm{P}(\mathbf{Nat}) = \{(\lambda_n) \mid \sum_n \lambda_n \leq 1\})$

**Equipped with a structure of comonoid in the *linear* Pcoh:**

- Cocontraction: $c^{\mathtt{nat}} : \mathtt{nat} \to \mathtt{nat} \otimes \mathtt{nat}$
- Coweakening: $w^{\mathtt{nat}} : \mathtt{nat} \to \mathbf{1}$

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# What sem. object to encode Storage Operator.

**The Eilenberg Moore Category: $\mathbf{Pcoh}^!$**

Coalgebras $P = (\underline{P}, h_P)$ with $\underline{P} \in \mathbf{Pcoh}$ and $h_P \in \mathbf{Pcoh}(\underline{P}, !\underline{P})$:

$$\begin{tikzcd} \underline{P} \arrow[r, "h_P"] \arrow[dr, "\mathrm{Id}"'] & !\underline{P} \arrow[d, "\mathrm{der}_{\underline{P}}"] \\ & \underline{P} \end{tikzcd} \qquad \begin{tikzcd} \underline{P} \arrow[r, "h_P"] \arrow[d, "h_P"'] & !\underline{P} \arrow[d, "\mathrm{dig}_{\underline{P}}"] \\ !\underline{P} \arrow[r, "!h_P"] & !!\underline{P} \end{tikzcd}$$

**Coalgebras** have a comonoid structure: values can be **stored**.

**Types interpreted as coalgebras:**

$!X$ by def. of the exp. | $\otimes$, $\oplus$ and fix preserve coalgebras.

**Example**

**Stream:** $\mathsf{S}_\varphi = \varphi \otimes !\mathsf{S}_\varphi$ | **List:** $\lambda_0 = \mathbf{1} \oplus (\varphi \otimes \lambda_0)$

# Probabilistic Call By Push Value

## Types:

**(Value)** $A \;::=\; U\underline{B} \mid A_1 \oplus A_2 \mid \mathbf{1} \mid A_1 \otimes A_2 \mid \alpha \mid \mathsf{Fix}\,\alpha \cdot A$

Example of natural numbers: $\mathtt{nat} ::= \mathsf{Fix}\,\alpha \cdot \mathbf{1} \oplus \alpha$

**(Computation)** $\underline{B} \;::=\; FA \mid A \multimap \underline{B}$

## Terms:

**(Value)** $V \;::=\; x \mid \mathsf{thunk}(M) \mid \mathsf{in}_i V \mid () \mid (V, W)$

**(Computation)** $M \;::=\; \mathsf{return}(V) \mid \mathsf{force}(M)$

$\mid \lambda x^A\, M \mid \langle M \rangle V \mid \mathbf{fix}(M)$

$\mid \mathsf{coin} \mid \mathsf{case}(M, x_1 \cdot N_1, x_2 \cdot N_2)$

$\mid \underline{n} \mid \mathsf{succ}(V) \mid \mathtt{let}\, x{=}V \,\mathtt{in}\, M \mid \mathtt{ifz}(V, M, N)$

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$\mid \lambda x^A\, M \mid \langle M \rangle V \mid \mathbf{fix}(M)$

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# Probabilistic Call By Push Value

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**(Value)** $A \ ::= \ U\underline{B} \mid A_1 \oplus A_2 \mid \mathbf{1} \mid A_1 \otimes A_2 \mid \alpha \mid \mathrm{Fix}\,\alpha \cdot A$

Example of natural numbers: $\mathtt{nat} ::= \mathrm{Fix}\,\alpha \cdot \mathbf{1} \oplus \alpha$

**(Computation)** $\underline{B} \ ::= \ FA \mid A \multimap \underline{B}$ Forget: $A$

**Terms:** $M^!$ $\mathrm{der}(M)$

**(Value)** $V \ ::= \ x \mid \mathrm{thunk}(M) \mid \mathrm{in}_i V \mid () \mid (V, W)$

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# The Eilenberg Moore categoy and the Linear Category

**Dense coalgebra**

$P = (\underline{P}, h_P)$ such that coalgebraic points characterize morphisms:
$\forall Y \in \mathbf{Pcoh}$ and $\forall t, t' \in \mathbf{Pcoh}(\underline{P}, Y)$,
if $\forall v \in \mathbf{Pcoh}^!(1, P),\ t\, v = t'\, v$, then $\forall u \in \mathbf{Pcoh}(1, \underline{P}),\ t\, u = t'\, u$.

Already known for $!X$ as: if $\forall x \in \mathbf{Pcoh}(1, X),\ t\, x^! = t'\, x^!$ then $t = t'$.

**The Eilenberg Moore category $\mathbf{Pcoh}^!$**

**Value Types** are interpreted as **dense** coalgebras

**Values** are morphisms of coalgebras

**The Linear category $\mathbf{Pcoh}$**

**Computation Types** are interpreted in $\mathbf{Pcoh}$

**Computations** are linear morphisms in $\mathbf{Pcoh}$

# Probabilistic Full Abstraction

**Theorem (2016: Ehrhard - T.)**

| **Pcoh** | | **pCBPV** |
|---|---|---|
| $[\![M]\!] = [\![N]\!]$ | Adequacy $\Longrightarrow$ $\Longleftarrow$ Full Abstraction | $M \simeq_o N$ |
| | | $\mathbf{Prob}(C[M], ()) \overset{\forall C[\,]}{=} \mathbf{Prob}(C[N], ())$ |

**Adequacy Lemma Proof:**

- Handle **values** separately
- Logical relations: **fixpoint** of types (hidden step indexing, biorthogonality closure, fixpoints of pairs of logical relations)
- **Density:** Morphisms on positive types are characterized by their action on coalgebraic points.

**Theorem (2016: Ehrhard - T.)**

| **Pcoh** | | **pCBPV** |
|---|---|---|
| $[\![M]\!] = [\![N]\!]$ | Adequacy $\Longrightarrow$ / $\Longleftarrow$ Full Abstraction | $M \simeq_o N$ |
| | | $\mathbf{Prob}(C[M],()) \overset{\forall C[\,]}{=} \mathbf{Prob}(C[N],())$ |

## Full Abstraction Proof:

1. By **contradiction**: $\exists \alpha \in |\sigma|,\ [\![M]\!]_\alpha \neq [\![N]\!]_\alpha$
2. Find **testing context**: $T_\alpha$ such that $[\![\langle T_\alpha \rangle M^!]\!] \neq [\![\langle T_\alpha \rangle N^!]\!]$ (context only depends on $\alpha$)
3. Prove **definability**: $T_\alpha \in$ **pCBPV** using coin and regularity of analytic functions and density.
4. Apply **Adequacy Lemma**:
$\mathbf{Prob}(\langle T_\alpha \rangle M^! \xrightarrow{*} ()) \neq \mathbf{Prob}(\langle T_\alpha \rangle N^! \xrightarrow{*} ())$.

**A Cone** $P$ is analogous to a real normed vector space, except that scalars are $\mathbb{R}^+$ and the norm $\|\_\|_P : P \to \mathbb{R}^+$ satisfies:

$$x + y = 0 \Rightarrow x, y = 0, \qquad \|x + x'\|_P \leq \|x\|_P + \|x'\|_P, \qquad \|\alpha x\|_P = \alpha\|x\|_P$$
$$x + y = x + y' \Rightarrow y = y', \qquad \|x\|_P = 0 \Rightarrow x = 0, \qquad \|x\|_P \leq \|x + x'\|_P$$

**The Unit Ball** is the set $\mathcal{B}P = \{x \in P \mid \|x\|_P \leq 1\}$.

**Order** $x \leq_P x'$ if there is a $y \in P$ such that $x' = x + y$. This unique $y$ is denoted as $y = x' - x$.

**A Complete Cone** is s.t. any non-decreasing $(x_n)_{n\in\mathbb{N}}$ of $\mathcal{B}P$ has a lub and $\|\sup_{n\in\mathbb{N}} x_n\|_P = \sup_{n\in\mathbb{N}} \|x_n\|_P$.

**Example of Complete Cones**

- $\mathrm{Meas}(X)$ with $X$ a measurable space.
- $\widehat{\mathcal{X}} = \{u \in (\mathbb{R}^+)^{|\mathcal{X}|} \mid \exists\varepsilon > 0\ \varepsilon u \in \mathrm{P}\mathcal{X}\}$ if $\mathcal{X} \in \mathbf{Pcoh}$.

The category of **complete cones** and **Scott-continuous** functions is not cartesian closed as *currying* fails to be *non-decreasing*.

A function $f : \mathcal{B}P \to Q$ is **n-non-decreasing function** if:

$n = 0$ and $f$ is non-decreasing

$n > 0$ and $\forall u \in \mathcal{B}P$, $\Delta f(x; u) = f(x+u) - f(x)$ is $(n-1)$-non-decreasing in $x$.

A function is **stable** if it is Scott-continuous and $\infty$-non-decreasing, i.e. $n$-non-decreasing for all $n \in \mathbb{N}$.

**Complete cones** and **stable** functions constitute a **CCC**.

**Weak Parallel Or**

wpor : $[0,1] \times [0,1] \to [0,1]$ given as $\mathrm{wpor}(s,t) = s + t - st$ is Scott-continuous, but not Stable. Its currying is not Scott-continuous.

**Type** $\texttt{real}$ is interpreted as $[\![\texttt{real}]\!] = \mathrm{Meas}(\mathbb{R})$,
**Closed term** $\vdash M : \texttt{real}$ as a measure $\mu$ and
**Term** $x : \texttt{real} \vdash N : \texttt{real}$ as a stable $f : \mathrm{Meas}(\mathbb{R}) \to \mathrm{Meas}(\mathbb{R})$.

**Operational semantics**

$$\forall r,\ \text{s.t. } M \to r,\ \texttt{let}\, x{=}M\, \texttt{in}\, N \to N\{r/x\}$$

By **Soundness**

$$[\![\texttt{let}\, x{=}M\, \texttt{in}\, N]\!] = \int_{\mathbb{R}} (\ f\ \circ\ \delta\ )(r)\ \ \mu\ (dr)$$

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$$[\![N]\!]$$

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## **Step 3:** The Measurability Problem

**Type** `real` is interpreted as $[\![\texttt{real}]\!] = \mathrm{Meas}(\mathbb{R})$,
**Closed term** $\vdash M : \texttt{real}$ as a measure $\mu$ and
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**Operational semantics**

$$\forall r, \text{ s.t. } M \to r, \ \texttt{let}\, x{=}M\, \texttt{in}\, N \to N\{r/x\}$$

By **Soundness**

$$[\![\texttt{let}\, x{=}M\, \texttt{in}\, N]\!] = \int_{\mathbb{R}} (\underbrace{f}_{[\![N]\!]} \circ \underbrace{\delta}_{\text{Dirac measure}})(r) \ \underbrace{\mu}_{[\![M]\!]}(dr)$$

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**Term** $x : \mathtt{real} \vdash N : \mathtt{real}$ as a stable $f : \mathrm{Meas}(\mathbb{R}) \to \mathrm{Meas}(\mathbb{R})$.

**Operational semantics**

$$\forall r, \text{ s.t. } M \to r, \ \mathtt{let}\, x{=}M\, \mathtt{in}\, N \to N\{r/x\}$$

By **Soundness**

$$[\![\mathtt{let}\, x{=}M\, \mathtt{in}\, N]\!] = \int_{\mathbb{R}} (\ f \ \circ \ \delta\ )(r) \ \ \mu\ (dr)$$

**Thus $f \circ \delta$ needs to be measurable.**

- There are non measurable stable functions
- We need to equip every cone with a notion of measurability

**Measurability tests of $\mathrm{Meas}(\mathbb{R})$ are given by measurable sets of $\mathbb{R}$:**

$$\forall U \subseteq \mathbb{R} \text{ measurable}, \ \varepsilon_U \in \mathrm{Meas}(\mathbb{R})' : \mu \mapsto \mu(U)$$

For needs of CCC, we parameterized measurable tests of a cone:

**Measurable Cone**

A cone $P$ with a collection $(\mathrm{M}^n(P))_{n \in \mathbb{N}}$ with $\mathrm{M}^n(P) \subseteq (P')^{\mathbb{R}^n}$ s.t.:

$$0 \in \mathrm{M}^n(P), \quad \ell \in \mathrm{M}^n(P) \text{ and } h : \mathbb{R}^p \to \mathbb{R}^n \Rightarrow \ell \circ h \in \mathrm{M}^p(P)$$

$$\ell \in \mathrm{M}^n(P) \text{ and } x \in P \Rightarrow \left\{ \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^+ \\ \vec{r} & \mapsto & \ell(\vec{r})(x) \end{array} \right. \text{ measurable.}$$

$\mathbf{Cstab_m}$ is the category of complete and measurable cones with stable and measurable functions.

Let $P$ and $Q$ be measurable and complete cones:

Measurable Test: $\mathsf{M}^n(P) \subseteq (P')^{\mathbb{R}^n}$

Measurable Path: $\mathsf{Path}^n(P) \subseteq P^{\mathbb{R}^n}$ the set of bounded $\gamma : \mathbb{R}^n \to P$ such that $\ell * \gamma : \mathbb{R}^{k+n} \to \mathbb{R}^+$ is measurable with

$$\ell * \gamma : (\vec{r}, \vec{s}) \mapsto \ell(\vec{r})(\gamma(\vec{s}))$$

Measurable Functions: Stable functions $f : P \to Q$ such that:

$$\forall n \in \mathbb{N},\ \forall \gamma \in \mathsf{Path}^n_1(P), \quad f \circ \gamma \in \mathsf{Path}^n(Q)$$

If $X$ is a measurable space, then $\mathsf{Meas}(X)$ is equipped with:
$\mathsf{M}^n(X) = \{\varepsilon_U : \mathbb{R}^n \to \mathsf{Meas}(X)' \text{ s.t. } \varepsilon_U(\vec{r})(\mu) = \mu(U),\ U \text{ meas.}\}$
$\mathsf{Path}^n_1(P)$ is the set of stochastic kernels from $\mathbb{R}^n$ to $X$.