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Quantitative Semantics for Probabilistic Programing

joint work with **T. Ehrhard** and **M. Pagani**

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A denotational semantics for probabilistic higher-order functional computation,

(based on **quantitative** semantics of Linear Logic)

Discrete setting:

Probabilistic Coherent Spaces are **fully abstract** for a programming language with **natural numbers** as base types suitable to encode discrete probabilistic programs.

Continuous setting:

A **CCC** of measurable spaces and **stable** maps that soundly denotes a programming language with **reals** as base types suitable to encode continuous probabilistic programs.

1. **Discrete Probability**
    - Syntax: **Discrete** Probabilistic PCF
    - Semantics: **Pcoh** (Probabilistic Coherent Spaces)
    - Results: Probabilistic **Adequacy** & **Full Abstraction**
    - Discrete Probabilistic Call By Push Value
2. **Continuous Probability**

General Framework	**Domains Semantics**	**Quantitative Semantics**
Types	Continuous **dcpos** $(X, \leq)$	**Proba.** spaces $(\lvert X \rvert, \mathrm{P}(X) \subseteq (\mathbb{R}^+)^{\lvert X \rvert})$
Programs	**Scott Continuous**	**Analytic** Functions
Probability	Proba. **monad**	**Values** as proba. distr.

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General Framework	**Domains Semantics**	**Quantitative Semantics**
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How to interpret a program $M : \mathcal{N} \Rightarrow \mathcal{N}$

Type:

$\mathbb{N}_\perp$ flat domain,
$\mathcal{V}(\mathbb{N}_\perp)$ proba. distr. over $\mathbb{N}_\perp$,

Prog: $[\![M]\!] : \mathbb{N}_\perp \to \mathcal{V}(\mathbb{N}_\perp)$,
$[\![$`let n=x in M`$]\!]$: $\mathcal{V}(\mathbb{N}_\perp) \to \mathcal{V}(\mathbb{N}_\perp)$

$$x \mapsto \left(\sum_n [\![M]\!]_{n,q} x_n \right)_q$$

Type:

$|\textbf{Nat}| = \mathbb{N}$
$\mathrm{P}(\textbf{Nat})$ subproba. dist. over $\mathbb{N}$

Prog: $[\![M]\!] : \mathrm{P}(\textbf{Nat}) \to \mathrm{P}(\textbf{Nat})$

$$x \mapsto \left(\sum_{\mu = [n_1, \ldots, n_k]} [\![M]\!]_{\mu,q} \prod_{i=1}^{k} x_{n_i} \right)_q$$

General Framework	**Domains Semantics**	**Quantitative Semantics**
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Problematic in **domain**

Finding a full subcategory of continuous dcpos that is: **Cartesian Closed** and **closed** under the proba. monad $\mathcal{V}$.

Full Abs.: **PCOH/pPCF**

$$\mathrm{Red}(C[M], \underline{n}) \overset{\forall n,\ \forall C[]}{=} \mathrm{Red}(C[N], \underline{n})$$

iff

$$[\![M]\!] = [\![N]\!].$$

1. **Discrete Probability**
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2. **Continuous Probability**

Syntax of PPCF

Syntax of PPCF:

Types: $A, B ::= \mathcal{N} \mid A \to B$

Terms: $M, N, L ::= x \mid \lambda x^A.M \mid (M)N \mid \mathtt{Y}M \mid$
$\mathsf{coin} \mid \underline{n} \mid \mathsf{succ}(M) \mid \mathtt{ifz}(L, M, N)$

Operational Semantics:

$\mathrm{Red}(M, N)$ is the **probability** that M reduces to N in a step.

$$\mathrm{Red}((\lambda x^A.M)N, M[N/x]) = 1, \text{ as } (\lambda x^A.M)N \xrightarrow{1} M[N/x]$$

$$\mathrm{Red}(\mathsf{coin}, \underline{0}) = \mathrm{Red}(\mathsf{coin}, \underline{1}) = \tfrac{1}{2}, \text{ as } \begin{tikzcd} & 0 \\ \mathsf{coin} \arrow[ru, "\frac{1}{2}"] \arrow[rd] & \\ & 1 \end{tikzcd}$$

If $\vdash M : \mathcal{N}$, then $\mathrm{Red}^\infty(M, \_)$ is the discrete distribution over $\mathbb{N}$ of all normal forms computed by M.

1 **Discrete Probability**

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2 **Continuous Probability**

Types as **Probabilistic Coherent Spaces**: $(|X|, P(X))$

Proba. Space

$|X|$: the **web**, a (potentially infinite) set of final states

$\mathrm{P}(X)$: a set of vectors $\subseteq (\mathbb{R}^+)^{|X|}$ such that

closure: $\mathbf{P}(\mathbf{X})^{\perp\perp} = \mathbf{P}(\mathbf{X})$ with

$$\forall u, v \in (\mathbb{R}^+)^{|X|},\ \langle u, v\rangle = \sum_{a\in|X|} u_a v_a$$

$$\forall P \subseteq (\mathbb{R}^+)^{|X|},\ P^\perp = \{v \in (\mathbb{R}^+)^{|X|}\ ;\ \forall u \in P,\ \langle u, v\rangle \leq 1\}$$

bounded covering: $\forall a \in |X|$,

$$\exists v \in P(X)\ ;\ v_a \neq 0 \quad \text{and} \quad \exists p > 0,\ ;\ \forall v \in \mathrm{P}(X),\ v_a \leq p.$$

Proposition: Proba. spaces as Domains

$(|X|, \mathrm{P}(X))$ is a **Proba. space iff** $\mathrm{P}(X)$ is bounded covering, **Scott Closed** (downwards-closed and dcpo) and **Convex**.

Types as **Probabilistic Coherent Spaces**: $(|X|, P(X))$

Example: $P(X) \subseteq (\mathbb{R}^+)^{|X|}$

$$|\mathbf{1}| = \{*\} \qquad \mathrm{P}\,(\mathbf{1}) = [0,1]$$

$$|\mathbf{Bool}| = \{t, f\} \qquad \mathrm{P}\,(\mathbf{Bool}) = \{(x_t, x_f)\ ;\ x_t + x_f \leq 1\}$$

$$|\mathbf{Nat}| = \{0, 1, 2, \dots\} \qquad \mathrm{P}\,(\mathbf{Nat}) = \{x \in [0,1]^{\mathbb{N}}\ ;\ \sum_n x_n \leq 1\}$$

$$|\mathbf{Bool} \to \mathbf{1}| = \{[t^n, f^m]\ ;\ n, m \in \mathbb{N}\},$$

$$\mathrm{P}\,(\mathbf{Bool} \to \mathbf{1}) = \{Q \in (\mathbb{R}^+)^{|\mathbf{Bool} \to \mathbf{1}|}\ ;$$

$$\forall x_t + x_f \leq 1, \quad \sum_{n,m=0}^{\infty} Q_{[t^n, f^m]}\, x_t^n x_f^m \leq 1\}$$

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$(|X|, \mathrm{P}\,(X))$ is a **Proba. space iff** $\mathrm{P}\,(X)$ is bounded covering, **Scott Closed** (downwards-closed and dcpo) and **Convex**.

A model of Linear Logic

Pcoh : Linear Category

Objects: Proba. Spaces

Morphisms: Linear Functions

- **Smcc** $(\mathbf{1}, \otimes, \multimap)$
- biproduct

Call by Name $A \to B = !A \multimap B$

- **Comonad** $(!, \mathrm{der}, \mathrm{dig})$
- **Com. Comonoid** $(!A, \mathbf{1}, \otimes)$

$\mathbf{Pcoh}_!$: Kleisli Category

Objects: Proba. Spaces

Morphisms: Analytic Functions

- **CCC**
- (PCF+coin)

Linear Category

$\mathbf{Pcoh}(X, Y)$

Matrices $Q \in (\mathbb{R}^+)^{|X| \times |Y|}$ such that:

$$\forall x \in \mathrm{P}(X),\ Q \cdot x = \left(\sum_{a \in |X|} Q_{a,b}\, x_a \right)_b \in \mathrm{P}(Y)$$

Example

$\mathbf{Pcoh}(\mathbf{Nat}, \mathbf{Nat})$: Stochastic Matrices $Q \in (\mathbb{R}^+)^{\mathbb{N} \times \mathbb{N}}$.

$$\forall x \in (\mathbb{R}^+)^{\mathbb{N}} \ ; \ \sum_{n \in \mathbb{N}} x_n \leq 1, \ \sum_{m,n \in \mathbb{N}} Q_{m,n} x_n \leq 1$$

Free Commutative Comonoid and Comonad

Exponential

$$|!X| = \mathcal{M}_{\mathsf{fin}}(|X|) \text{ the set of finite multisets}$$

$$\mathrm{P}(!X) = \{x^! \;;\; x \in \mathrm{P}(X)\}^{\perp\perp} \text{ where } x^!_{[a_1,\ldots,a_k]} = \prod_{i=1}^{k} x_{a_i}$$

Example

Let **Bcoin** $= (p, 1-p) \in \mathrm{P}(\mathbf{Bool}) = \{(p,q) \;;\; p+q \leq 1\}$.

$$\mathbf{Bcoin}^!_{[]} = 1, \qquad \mathbf{Bcoin}^!_{[t,t]} = p^2, \qquad \mathbf{Bcoin}^!_{[t,f]} = p(1-p), \;\ldots$$

Theorem (2017: Crubillé - Ehrhard - Pagani - T.)

This exponential computes the free commutative comonoid.

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Commutative Comonoid | **Comonad**

Cocontr.: $!X \xrightarrow{c^{!X}} !X \otimes !X$

Coweak.: $!X \xrightarrow{w^{!X}} \mathbf{1}$

Comult.: $\mathrm{dig}_{!X} : !!X \to !X$

Counit: $\mathrm{der}_{!X} : !X \to X$

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Non Linear Category

$\mathbf{Pcoh}_!(X, Y) = \mathbf{Pcoh}(!X, Y)$

Matrices $Q \in (\mathbb{R}^+)^{\mathcal{M}_{\mathrm{fin}}(|X|)\times|Y|}$ such that

$$\forall U \in \mathrm{P}\,(!X),\ \ Q\cdot U = \left(\sum_{m\in\mathcal{M}_{\mathrm{fin}}(|X|)} Q_{m,b}\, U_m\right)_b \in \mathrm{P}\,(Y)$$

Non-Linear Morphisms are **analytic** and **Scott Continuous**.

$$\mathbf{Pcoh}_!(\mathbf{Bool},\mathbf{1}) = \{Q \in (\mathbb{R}^+)^{|\mathbf{Bool}\to\mathbf{1}|}\ \ s.t.\ \ Q_{[t^n,f^m]} \le \frac{(n+m)^{n+m}}{n^n\, m^m}\}$$

```
let rec f x =
 if x then if x then f x
                else ()
      else if x then ()
                else f x
```

denotes

$$\sum_{n,m=0}^{\infty} \frac{(n+m)!}{n!\, m!} x_t^{2n+1} x_f^{2m+1}$$

Non Linear Category

$\mathbf{Pcoh}_!(X, Y) = \mathbf{Pcoh}(!X, Y)$ **Density**

Matrices $Q \in (\mathbb{R}^+)^{\mathcal{M}_{\text{fin}}(|X|)\times|Y|}$ such that if $x^!_m = \prod_{a\in m} x_a^{m(a)}$

$$\forall x \in \mathrm{P}\,(X),\ \ Q(x) = \left(\sum_{m\in\mathcal{M}_{\text{fin}}(|X|)} Q_{m,b}\, {x^!}_m\right)_b \in \mathrm{P}\,(Y)$$

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pb of DEFINABILITY

$$\sum_{n,m=0}^{\infty} \frac{(n+m)!}{n!\,m!} x_t^{2n+1} x_f^{2m+1}$$

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Probabilistic Full Abstraction

Theorem (2014: Ehrhard - Pagani - T.)

Pcoh

$[\![M]\!] = [\![N]\!]$

Adequacy

$\Longrightarrow$

$\Longleftarrow$

Full Abstraction

pPCF

$M \simeq_o N$

$\mathrm{Red}^\infty(C[M], n) \overset{\forall C[\,]\forall n}{=} \mathrm{Red}^\infty(C[N], n)$

Adequacy Lemma (2011: Danos - Ehrhard):

If $\vdash M : \mathcal{N}$, then $\forall n \in \mathbb{N}, [\![M]\!]_n = \mathrm{Red}^\infty(M, n)$.

Adequacy proof:

If $[\![M]\!] = [\![N]\!]$ then, $\mathrm{Red}^\infty((C)M, \underline{n}) = \mathrm{Red}^\infty((C)N, \underline{n})$

1. Apply Adequacy Lemma : $\mathrm{Red}^\infty((C)M, \underline{n}) = [\![(C)M]\!]$.
2. Apply Compositionality:

$$[\![(C)M]\!] = \sum_\mu [\![C]\!]_\mu \prod_{\alpha \in \mu} [\![M]\!]_\alpha^{\mu(\alpha)} = \sum_\mu [\![C]\!]_\mu \prod_{\alpha \in \mu} [\![N]\!]_\alpha^{\mu(\alpha)} = [\![(C)N]\!]$$

Probabilistic Full Abstraction

Theorem (2014: Ehrhard - Pagani - T.)

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$[\![M]\!] = [\![N]\!]$

Adequacy

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Adequacy Lemma (2011: Danos - Ehrhard):

If $\vdash M : \mathcal{N}$, then $\forall n \in \mathbb{N}, [\![M]\!]_n = \mathrm{Red}^\infty(M, n)$.

Full Abstraction proof:

- Find **testing terms** that depend only on points of the web.
- Use regularity of **analytic functions**.

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2. **Continuous Probability**

How to encode a LasVegas Algorithm?

Input: A $\underline{0}/\underline{1}$ array of length $n \geq 2$ s.t. $\frac{1}{2}$ cells are $\underline{0}$.

0	1	2	3	4	5
$\underline{0}$	$\underline{1}$	$\underline{0}$	$\underline{1}$	$\underline{1}$	$\underline{0}$

$$f : \begin{array}{ccc} 0, 2, 5 & \mapsto & \underline{0} \\ 1, 3, 4 & \mapsto & \underline{1} \end{array}$$

Output: Find the index of a cell containing $\underline{0}$.

Caml:

```
let rec LasVegas (f: nat -> nat) (n:nat) =
                let k = random n in
                    if (f k = 0) then k
                    else LasVegas f n
```

pPCF: CBN

$$\mathbf{Y}\Big(\lambda \mathtt{LasVegas}^{(\mathtt{nat}\Rightarrow\mathtt{nat})\Rightarrow\mathtt{nat}\Rightarrow\mathtt{nat}}\ \lambda \mathtt{f}^{\mathtt{nat}\Rightarrow\mathtt{nat}} \lambda \mathtt{n}^{\mathtt{nat}}$$
$$(\lambda \mathtt{k}^{\mathtt{nat}}\ \mathtt{ifz}\ \mathtt{f}\ \mathtt{k}\ \mathtt{then}\ \mathtt{k}$$
$$\mathtt{else}\ \mathtt{LasVegas}\ \mathtt{f}\ \mathtt{n})\ (\mathtt{rand}\ \mathtt{n})$$

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pPCF:
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$$\mathbf{Y}\Big(\lambda \mathtt{LasVegas}^{(\mathtt{nat}\Rightarrow\mathtt{nat})\Rightarrow\mathtt{nat}\Rightarrow\mathtt{nat}}\ \lambda \mathtt{f}^{\mathtt{nat}\Rightarrow\mathtt{nat}}\lambda \mathtt{n}^{\mathtt{nat}}$$
$$(\lambda \mathtt{k}^{\mathtt{nat}}\ \mathtt{ifz\ f\ k\ then\ k}$$
$$\mathtt{else\ LasVegas\ f\ n})\ (\mathtt{rand\ n})\Big)$$

Semantics gives the answer

Storage Operator

```
let k = rand n in if k = 0 then k else 42
```

Integer in Pcoh: $[\![\mathcal{N}]\!] = \mathbf{Nat} = (\mathbb{N}, \mathrm{P}(\mathbf{Nat}) = \{(\lambda_n) \mid \sum_n \lambda_n \leq 1\})$

Equipped with a structure of comonoid in the *linear* Pcoh:

- Cocontraction: $c^{\mathcal{N}} : \mathcal{N} \to \mathcal{N} \otimes \mathcal{N}$
- Coweakening: $w^{\mathcal{N}} : \mathcal{N} \to \mathbf{1}$

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What sem. object to encode Storage Operator.

The Eilenberg Moore Category: $\mathbf{Pcoh}^!$

Coalgebras $P = (\underline{P}, h_P)$ with $\underline{P} \in \mathbf{Pcoh}$ and $h_P \in \mathbf{Pcoh}(\underline{P}, !\underline{P})$:

$$\begin{tikzcd}
\underline{P} \arrow[r, "h_P"] \arrow[dr, "\mathrm{Id}"'] & !\underline{P} \arrow[d, "\mathrm{der}_{\underline{P}}"] \\
 & \underline{P}
\end{tikzcd}
\qquad
\begin{tikzcd}
\underline{P} \arrow[r, "h_P"] \arrow[d, "h_P"'] & !\underline{P} \arrow[d, "\mathrm{dig}_{\underline{P}}"] \\
!\underline{P} \arrow[r, "!h_P"] & !!\underline{P}
\end{tikzcd}$$

Coalgebras have a comonoid structure: values can be **stored**.

Types interpreted as coalgebras:

$!X$ by def. of the exp. | $\otimes$, $\oplus$ and Y preserve coalgebras.

Example

Stream: $\mathsf{S}_\varphi = \varphi \otimes !\mathsf{S}_\varphi$ | **List:** $\lambda_0 = \mathbf{1} \oplus (\varphi \otimes \lambda_0)$

Probabilistic Call By Push Value

Types:

(Value) $A ::= U\underline{B} \mid A_1 \oplus A_2 \mid \mathbf{1} \mid A_1 \otimes A_2 \mid \alpha \mid \mathrm{Fix}\,\alpha \cdot A$

Example of natural numbers: $\mathcal{N} ::= \mathrm{Fix}\,\alpha \cdot \mathbf{1} \oplus \alpha$

(Computation) $\underline{B} ::= FA \mid A \multimap \underline{B}$

Terms:

(Value) $V ::= x \mid \mathrm{thunk}(M) \mid \mathrm{in}_i V \mid () \mid (V, W)$

(Computation) $M ::= \mathrm{return}(V) \mid \mathrm{force}(M)$

$\mid \lambda x^A\, M \mid \langle M \rangle V \mid \Upsilon M$

$\mid \mathrm{coin} \mid \mathrm{case}(M, x_1 \cdot N_1, x_2 \cdot N_2)$

$\mid \underline{n} \mid \mathrm{succ}(V) \mid \mathtt{let}(x, V, M) \mid \mathtt{ifz}(V, M, N)$

Probabilistic Call By Push Value

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$\mid \lambda x^A M \mid \langle M \rangle V \mid \Upsilon M$

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Probabilistic Call By Push Value

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Terms: $M^!$ $\mathsf{der}(M)$

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The Eilenberg Moore categoy and the Linear Category

Dense coalgebra

$P = (\underline{P}, h_P)$ such that coalgebraic points characterize morphisms:
$\forall Y \in \mathbf{Pcoh}$ and $\forall t, t' \in \mathbf{Pcoh}(\underline{P}, Y)$,
if $\forall v \in \mathbf{Pcoh}^!(1, P),\ t\,v = t'\,v$, then $\forall u \in \mathbf{Pcoh}(1, \underline{P}),\ t\,u = t'\,u$.

Already known for $!X$ as: if $\forall x \in \mathbf{Pcoh}(1, X),\ t\,x^! = t'\,x^!$ then $t = t'$.

The Eilenberg Moore category $\mathbf{Pcoh}^!$

Value Types are interpreted as **dense** coalgebras

Values are morphisms of coalgebras

The Linear category **Pcoh**

Computation Types are interpreted in **Pcoh**

Computations are linear morphisms in **Pcoh**

Probabilistic Full Abstraction

Theorem (2016: Ehrhard - T.)

Pcoh

$[\![M]\!] = [\![N]\!]$

Adequacy

$\Longrightarrow$

$\Longleftarrow$

Full Abstraction

pCBPV

$M \simeq_o N$

$\mathrm{Red}(C[M], ()) \overset{\forall C[\,]}{=} \mathrm{Red}(C[N], ())$

Adequacy Lemma Proof:

- Handle **values** separately
- Logical relations: **fixpoint** of types (hidden step indexing, biorthogonality closure, fixpoints of pairs of logical relations)
- **Density:** Morphisms on positive types are characterized by their action on coalgebraic points.

Probabilistic Full Abstraction

Theorem (2016: Ehrhard - T.)

Pcoh

pCBPV

$[\![M]\!] = [\![N]\!]$ $\overset{\text{Adequacy}}{\Longrightarrow}$ $\underset{\text{Full Abstraction}}{\Longleftarrow}$ $M \simeq_o N$

$\mathrm{Red}(C[M], ()) \overset{\forall C[\,]}{=} \mathrm{Red}(C[N], ())$

Full Abstraction Proof:

1. By **contradiction**: $\exists \alpha \in |\sigma|,\ [\![M]\!]_\alpha \neq [\![N]\!]_\alpha$
2. Find **testing context**: T_α such that $[\![\langle T_\alpha\rangle M^!]\!] \neq [\![\langle T_\alpha\rangle N^!]\!]$ (context only depends on α)
3. Prove **definability**: $T_\alpha \in$ **pCBPV** using coin and regularity of analytic functions and density.
4. Apply **Adequacy Lemma**: $\mathrm{Red}(\langle T_\alpha\rangle M^! \xrightarrow{*} ()) \neq \mathrm{Red}(\langle T_\alpha\rangle N^! \xrightarrow{*} ())$.

A denotational semantics for probabilistic higher-order functional computation,

(based on **quantitative** semantics of Linear Logic)

Discrete setting:

Probabilistic Coherent Spaces are **fully abstract** for a programming language with **natural numbers** as base types suitable to encode discrete probabilistic programs.

Continuous setting:

A **CCC** of measurable spaces and **stable** maps that soundly denotes a programming language with **reals** as base types suitable to encode continuous probabilistic programs.

1. Discrete Probability

2. **Continuous Probability**
   - Syntax: **Real** Probabilistic PCF
   - Semantics: $\mathbf{Cstab}_{\mathrm{m}}$ (Cones and Stable measurable functions)
   - Results: **Adequacy**

From Discrete to Continuous syntax

PPCF

Types: $A, B ::= \mathcal{N} \mid A \to B$

Terms: $M, N, L ::=$

$x \mid \lambda x^{A}.M \mid (M)N \mid \mathtt{Y}M \mid$

$\underline{n} \mid \mathsf{succ}(M) \mid$

$\mathtt{ifz}(L, M, N) \mid$

$\mathsf{coin} \mid \mathtt{let}(x, M, N)$

Operational Semantics:

$\mathrm{Red}(\mathsf{coin}, \underline{0}) = \mathrm{Red}(\mathsf{coin}, \underline{1}) = \frac{1}{2}$

If $\vdash M : \mathcal{N}$, $\mathrm{Red}^{\infty}(M, \_)$ is the discrete distribution over $\mathbb{N}$ computed by M.

From Discrete to Continuous syntax

PPCF

Types: $A, B ::= \mathcal{N} \mid A \to B$

Terms: $M, N, L ::=$
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Real PPCF

Types: $A, B ::= \mathcal{R} \mid A \to B$

Terms: $M, N, L ::=$
$x \mid \lambda x^A.M \mid (M)N \mid \mathtt{Y}M \mid$
$\underline{r} \mid \underline{f}(M_1, \ldots, M_n) \mid$
$\mathtt{ifz}(L, M, N) \mid$
$\mathtt{sample} \mid \mathtt{let}(x, M, N)$

Operational Semantics:

$\mathrm{Red}(\mathtt{sample}, U) = \lambda_{[0,1]}(U)$

If $\vdash M : \mathcal{R}$, $\mathrm{Red}^\infty(M, \_)$ is the continuous distribution over $\mathbb{R}$ computed by M.

Operational Semantics

The probability to observe U after at most one reduction step applied to M is $\mathrm{Red}(\ M\ ,\ U\ )$

$\mathrm{Red} : \Lambda^{\Gamma\vdash A} \times \Sigma_{\Lambda^{\Gamma\vdash A}} \to \mathbb{R}^+$ is a **Kernel**, i.e:

- for all $M \in \Lambda^{\Gamma\vdash A}$, $\mathrm{Red}(M, \_)$ is a measure;
- for all $U \in \Sigma_{\Lambda^{\Gamma\vdash A}}$, $\mathrm{Red}(\_, U)$ is a measurable function.

$\mathrm{Red}^{\infty}(M, U)$ is the probability to observe U after any steps.

Operational Semantics

The probability to observe U after at most one reduction step applied to M is $\mathrm{Red}(\ M\ ,\ U\ )$

$\Lambda^{\Gamma\vdash A}$: the set of terms M
s.t. $\Gamma \vdash M : A$.

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Measurable sets and kernels constitute the category Kern.

$\mathrm{Red}^{\infty}(M, U)$ is the probability to observe U after any steps.

Operational Semantics

The probability to observe U after at most one reduction step applied to M is $\mathrm{Red}(\ M\ ,\ U\ )$

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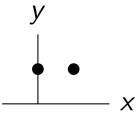
Measurable sets and kernels constitute the category Kern.

$\mathrm{Red}^{\infty}(M, U)$ is the probability to observe U after any steps.

It is computed by composition and lub.

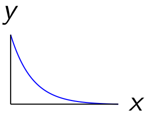
Examples: Distributions

The Bernoulli distribution takes the value 1 with probability p and the value 0 with probability $1-p$.



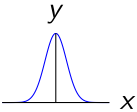
$\texttt{bernoulli}\ p\ ::=\ \texttt{let}(x, \texttt{sample}, x \leq p)$ tests if `sample` draws a value within $[0, p]$.

The exponential distribution is specified by its density e^{-x}.



$\texttt{exp} : \mathcal{R} ::= \texttt{let}(x, \texttt{sample}, -\log(x))$
by the inversion sampling method.

The standard normal distribution defined by its density $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.



$\texttt{normal} ::=$
$\texttt{let}(x, \texttt{sample}, \texttt{let}(y, \texttt{sample}, \sqrt{-2\log(x)}\cos(2\pi y)))$
by the Box Muller method.

Examples:

Conditioning: If $U \subseteq \mathbb{R}$ measurable, then $\texttt{observe}(U)$ of type $\mathcal{R} \to \mathcal{R}$, taking a term M and returning the renormalization of the distribution of M on the only samples that satisfy U:
$\texttt{observe}(U) = \lambda m.\texttt{Y}(\lambda y.\texttt{let}(x, m, \texttt{if}(x \in U, x, y)))$
conditioning by rejection sampling.

Monte Carlo Simulation, Metropolis Hasting,...

1. Discrete Probability

2. **Continuous Probability**
   - Syntax: **Real** Probabilistic PCF
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   - Results: **Adequacy**

Semantical context

1981, Kozen Memory as measurable space and programs as kernels representing the transformation of the memory.

What is a measurable subset for function space ?

1999, Panangaden

Meas, the category of measurable sets and functions

Kern, the category of measurable sets and kernels

They are **cartesian** but **not closed**.

2017, Heunen, Kammar, Staton, Yang **Quasi-borel spaces**

A **CCC** based on **Meas** embedded into presheaves.

How to interpret recursive types ?

2017, Keimel and Plotkin **Kegelspitzen**

A **CCC** of dcpos equipped with a convex structure (basic operations being scott continous) with scott continuous functions

How to restrict to measurable functions ?

Semantical context

If $\vdash M : \mathcal{N}$, then $[\![M]\!]$ is a discrete distribution over $\mathbb{N}$	If $\vdash M : \mathcal{R}$, then $[\![M]\!]$ is a continuous measure over $\mathbb{R}$

- $[\![\mathcal{R}]\!]$ as Meas($\mathbb{R}$) the set of measures over the measurable space $\mathbb{R}$.
- Fixpoint of terms.

Cstab$_{\mathrm{m}}$ is a **CCC** based on Selinger's **cones** (dcpos with the order induced by addition and a convex structure).

Objects are cones and measurable spaces

Morphisms are stable and measurable functions

Pcoh is a subcategory of **Cstab$_{\mathrm{m}}$** which is a subcategory of Kegelspitzen.

An elegant model in 3 steps

Our purpose is to be able to interpret $\mathcal{R}$ as the set of bounded measures.

1. **Complete cones** (convex dcpos with the order induced by addition) with Scott continuous functions
   However, the category is cartesian but not closed.
2. Complete cones and **Stable functions** (∞-non-decreasing functions) is a CCC.
   However, not every stable function is measurable.
3. **Measurable Cones** (complete cones with **measurable tests**). Measurable paths pass measurable tests and Measurable functions preserve measurable paths.
   $\mathbf{Cstab_m}$ is a CCC with measurability included !

Step 1: Complete Cones

A Cone P is analogous to a real normed vector space, except that scalars are $\mathbb{R}^+$ and the norm $\|\_\|_P : P \to \mathbb{R}^+$ satisfies:

$$x+y=0 \Rightarrow x,y=0, \qquad \|x+x'\|_P \leq \|x\|_P + \|x'\|_P, \qquad \|\alpha x\|_P = \alpha\|x\|_P$$

$$x+y=x+y' \Rightarrow y=y', \qquad \|x\|_P = 0 \Rightarrow x=0, \qquad \|x\|_P \leq \|x+x'\|_P$$

The Unit Ball is the set $\mathcal{B}P = \{x \in P \mid \|x\|_P \leq 1\}$.

Order $x \leq_P x'$ if there is a $y \in P$ such that $x' = x + y$. This unique y is denoted as $y = x' - x$.

A Complete Cone is s.t. any non-decreasing $(x_n)_{n\in\mathbb{N}}$ of $\mathcal{B}P$ has a lub and $\|\sup_{n\in\mathbb{N}} x_n\|_P = \sup_{n\in\mathbb{N}} \|x_n\|_P$.

Example of Complete Cones

- Meas(X) with X a measurable space.
- $\widehat{\mathcal{X}} = \{u \in (\mathbb{R}^+)^{|\mathcal{X}|} \mid \exists \varepsilon > 0\ \varepsilon u \in \mathrm{P}\mathcal{X}\}$ if $\mathcal{X} \in$ **Pcoh**.

Step 2: Stable functions

The category of **complete cones** and **Scott-continuous** functions is not cartesian closed as *currying* fails to be *non-decreasing*.

A function $f : \mathcal{B}P \to Q$ is **n-non-decreasing function** if:

$n = 0$ and f is non-decreasing

$n > 0$ and $\forall u \in \mathcal{B}P$, $\Delta f(x; u) = f(x + u) - f(x)$ is $(n-1)$-non-decreasing in x.

A function is **stable** if it is Scott-continuous and ∞-non-decreasing, i.e. n-non-decreasing for all $n \in \mathbb{N}$.

Complete cones and **stable** functions constitute a **CCC**.

Weak Parallel Or

$\mathrm{wpor} : [0,1] \times [0,1] \to [0,1]$ given as $\mathrm{wpor}(s,t) = s + t - st$ is Scott-continuous, but not Stable. Its currying is not Scott-continuous.

Step 3: The Measurability Problem

Type $\mathcal{R}$ is interpreted as $[\![\mathcal{R}]\!] = \mathrm{Meas}(\mathbb{R})$,
Closed term $\vdash M : \mathcal{R}$ as a measure μ and
Term $x : \mathcal{R} \vdash N : \mathcal{R}$ as a stable $f : \mathrm{Meas}(\mathbb{R}) \to \mathrm{Meas}(\mathbb{R})$.

Operational semantics

$$\forall r,\ \text{s.t. } M \to r,\ \mathtt{let}(x, M, N) \to N\{r/x\}$$

By **Soundness**

$$[\![\mathtt{let}(x, M, N)]\!] = \int_{\mathbb{R}} (\ f\ \circ\ \delta\ )(r)\ \ \mu\ (dr)$$

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By **Soundness**

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Step 3: The Measurability Problem

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Term $x : \mathcal{R} \vdash N : \mathcal{R}$ as a stable $f : \mathrm{Meas}(\mathbb{R}) \to \mathrm{Meas}(\mathbb{R})$.

Operational semantics

$$\forall r,\ \text{s.t. } M \to r,\ \texttt{let}(x, M, N) \to N\{r/x\}$$

By **Soundness**

$$[\![\texttt{let}(x, M, N)]\!] = \int_{\mathbb{R}} (\ f\ \circ\ \delta\ )(r)\ \ \mu\ (dr)$$

Thus $f \circ \delta$ needs to be measurable.

- There are non measurable stable functions
- We need to equip every cone with a notion of measurability

Step 3: Measurability tests

Measurability tests of $\mathrm{Meas}(\mathbb{R})$ are given by measurable sets of $\mathbb{R}$:

$$\forall U \subseteq \mathbb{R} \text{ measurable, } \varepsilon_U \in \mathrm{Meas}(\mathbb{R})' : \mu \mapsto \mu(U)$$

For needs of CCC, we parameterized measurable tests of a cone:

Measurable Cone

A cone P with a collection $(\mathrm{M}^n(P))_{n\in\mathbb{N}}$ with $\mathrm{M}^n(P) \subseteq (P')^{\mathbb{R}^n}$ s.t.:

$$0 \in \mathrm{M}^n(P), \quad \ell \in \mathrm{M}^n(P) \text{ and } h : \mathbb{R}^p \to \mathbb{R}^n \Rightarrow \ell \circ h \in \mathrm{M}^p(P)$$

$$\ell \in \mathrm{M}^n(P) \text{ and } x \in P \Rightarrow \left\{ \begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R}^+ \\ \vec{r} & \mapsto & \ell(\vec{r})(x) \end{array} \right. \text{ measurable.}$$

Measurable Tests, Paths and Functions

$\mathbf{Cstab}_{\mathsf{m}}$ is the category of complete and measurable cones with stable and measurable functions.

Let P and Q be measurable and complete cones:

Measurable Test: $\mathsf{M}^n(P) \subseteq (P')^{\mathbb{R}^n}$

Measurable Path: $\mathsf{Path}^n(P) \subseteq P^{\mathbb{R}^n}$ the set of bounded $\gamma : \mathbb{R}^n \to P$ such that $\ell * \gamma : \mathbb{R}^{k+n} \to \mathbb{R}^+$ is measurable with

$$\ell * \gamma : (\vec{r}, \vec{s}) \mapsto \ell(\vec{r})(\gamma(\vec{s}))$$

Measurable Functions: Stable functions $f : P \to Q$ such that:

$$\forall n \in \mathbb{N},\ \forall \gamma \in \mathsf{Path}_1^n(P), \quad f \circ \gamma \in \mathsf{Path}^n(Q)$$

If X is a measurable space, then $\mathsf{Meas}(X)$ is equipped with:
$\mathsf{M}^n(X) = \{\varepsilon_U : \mathbb{R}^n \to \mathsf{Meas}(X)' \text{ s.t. } \varepsilon_U(\vec{r})(\mu) = \mu(U),\ U \text{ meas.}\}$
$\mathsf{Path}_1^n(P)$ is the set of stochastic kernels from $\mathbb{R}^n$ to X.

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Results

The category **Cstab**$_{\mathrm{m}}$ is a CCC and a model of Real PPCF.

Interpretation of some terms:

$$[\![\underline{r}]\!] = \delta_r,\ [\![\texttt{sample}]\!] = \lambda_{[0,1]},\ [\![\texttt{let}(x, M, N)]\!](U) = \int_{\mathbb{R}} [\![N]\!](\delta_r)(U)\,[\![M]\!](dr)$$

Soundness

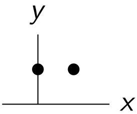
$$[\![M]\!]^{\Gamma\vdash A} = \int_{\Lambda^{\Gamma\vdash A}} [\![t]\!]^{\Gamma\vdash A}\mathrm{Red}(M, dt)$$

Adequacy

$$[\![M]\!]^{\vdash\mathcal{R}}(U) = \mathrm{Red}^{\infty}(M, U)$$

Examples: Distributions

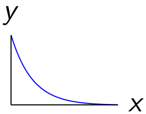
The Bernoulli distribution takes the value 1 with probability p and the value 0 with probability $1 - p$.



$$\texttt{bernoulli}\ p ::= \texttt{let}(x, \texttt{sample}, x \leq p)$$

$$\llbracket \texttt{bernoulli}\ \underline{p} \rrbracket^{\vdash \mathcal{R}} = p\delta_1 + (1-p)\delta_0$$

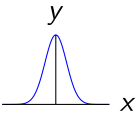
The exponential distribution is specified by its density e^{-x}.



$$\exp : \mathcal{R} ::= \texttt{let}(x, \texttt{sample}, -\log(x))$$

$$\llbracket \exp \rrbracket^{\vdash \mathcal{R}}(U) = \int_{\mathbb{R}^+} \chi_U(s) \mathrm{e}^{-s} \lambda(ds)$$

The standard normal distribution defined by its density $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$.



$$\texttt{normal} ::=$$

$$\texttt{let}(x, \texttt{sample}, \texttt{let}(y, \texttt{sample}, \sqrt{-2\log(x)}\cos(2\pi y)))$$

$$\llbracket \texttt{normal} \rrbracket^{\vdash \mathcal{R}}(U) = \frac{1}{\sqrt{2\pi}} \int_U \mathrm{e}^{-\frac{x^2}{2}} \lambda(dx)$$

Examples:

Conditioning: If $U \subseteq \mathbb{R}$ measurable, then $\texttt{observe}(U)$ of type $\mathcal{R} \to \mathcal{R}$, taking a term M and returning the renormalization of the distribution of M on the only samples that satisfy U:

$\texttt{observe}(U) = \lambda m.\texttt{Y}(\lambda y.\texttt{let}(x, m, \texttt{if}(x \in U, x, y)))$

conditioning by rejection sampling.

Whenever M represents a probability distribution, this equation gives the conditional probability:

$$\llbracket \texttt{observe}(U) M \rrbracket (V) = \frac{\llbracket M \rrbracket (V \cap U)}{\llbracket M \rrbracket (U)}$$

Conclusion

Pcoh and $\mathbf{Cstab_m}$ models of probabilistic programming

- For countable data types, **Pcoh** is fully abstract.
- For real data types, $\mathbf{Cstab_m}$ is a sound model that encodes probability measures used in probabilistic programming.

Further directions:

- A model of LL ?
- A model of pCBPV ?
- Full abstraction ?