Channel Synthesis Revisited

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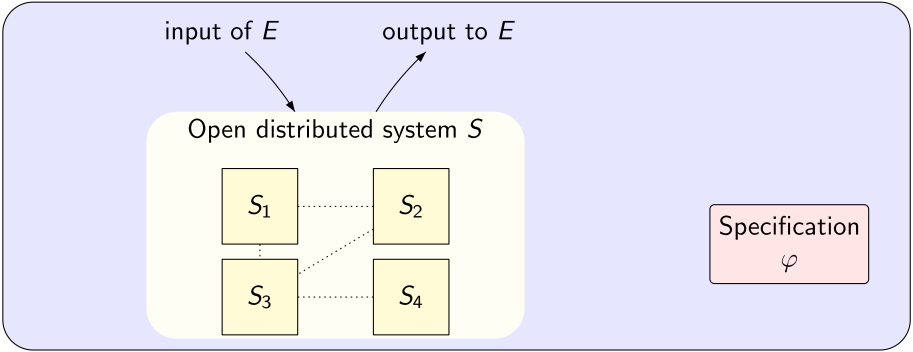
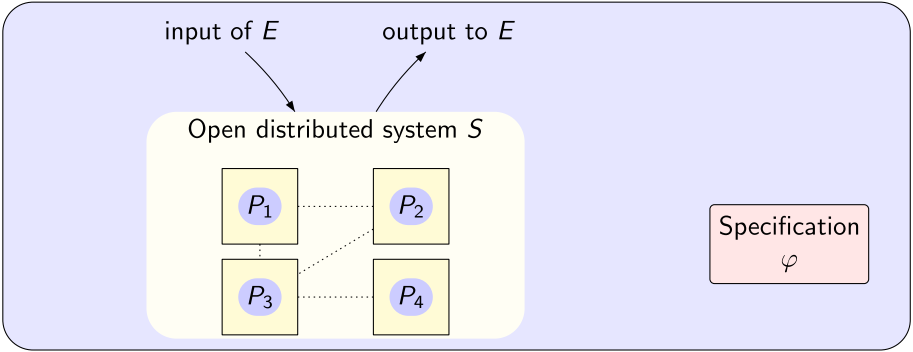
Distributed synthesis



Distributed synthesis



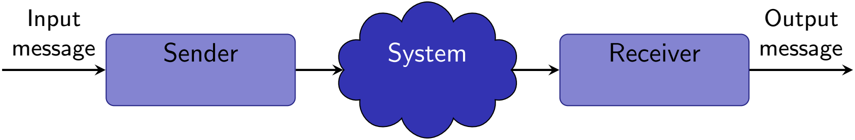
Problems

- Decide the existence of a distributed program such that the joint behavior $P_1||P_2||P_3||P_4||E$ satisfies φ, for all E.
- Synthesis: If it exists, compute such a distributed program.

$\rightsquigarrow$ Undecidable for asynchronous communication with two processes and total LTL specifications [Schewe, Finkbeiner; 2006].

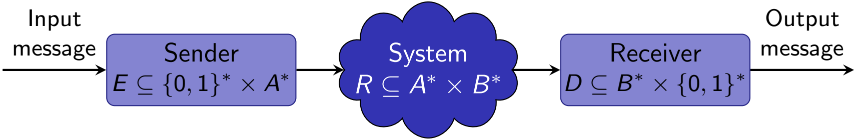
Channel synthesis

- Pipeline architecture with asynchronous transmission
- Simple external specification on finite binary messages:
  output message = input message (perfect data transmission)



Channel synthesis

- Pipeline architecture with asynchronous transmission
- Simple external specification on finite binary messages:
  output message = input message (perfect data transmission)
- All processes are finite transducers



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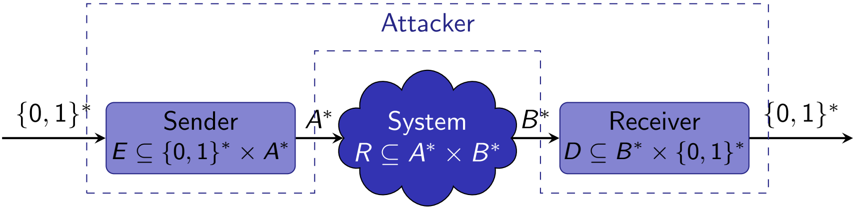
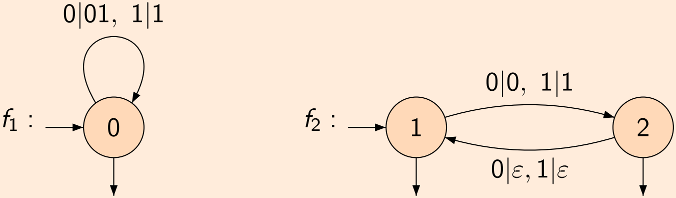


Finite transducers

A transducer is a finite automaton with set of labels $Lab \subseteq A^* \times B^*$,
it accepts a rational relation R,

- $dom(R) = \{u \in A^* \mid (u, v) \in R \text{ for some } v \in B^*\}$,
- $range(R) = \{v \in B^* \mid (u, v) \in R \text{ for some } u \in A^*\}$.

A relation R realized by a transducer, as a union of two functions: $R = f_1 + f_2$



$R(00) = \{0101, 0\}$ and $R(01) = \{011, 0\}$.

$range(f_1) = (01 + 1)^*$ and $range(f_2) = (0 + 1)^*$.

Rational channels

- The identity relation on A^* is $Id_{A^*} = \{(w, w) \mid w \in A^*\}$.
- Rational relations can be composed: RR'.

Definition

A channel for a rational relation R is a pair (E, D) of rational relations such that

$$ERD = Id_{\{0,1\}^*}$$

Problems:

Given R, does there exist a channel (E, D) for R? If it exists, can it be computed?

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Previous results [BBLMRS 2011]

- The channel synthesis problem is undecidable.
- When R is a function, the problem is decidable and if it exists, the channel can be computed in polynomial time.

Outline

Growth of languages

Patterns

Conclusion

Growth of languages

Definition

For a language L over alphabet A, the growth of L is δ_L:

$$\delta_L(n) = card(L \cap A^{\leq n})$$

L has polynomial growth if $\delta_L(n)$ is bounded by some polynomial
finite unions of languages of the form $u_1 v_1^* \ldots u_k v_k^* u_{k+1}$

L has exponential growth if $\delta_L(n)$ is greater than some exponential
languages containing some $u(v + \overline{v})^* w$ (where $\{v, \overline{v}\}$ is a code)

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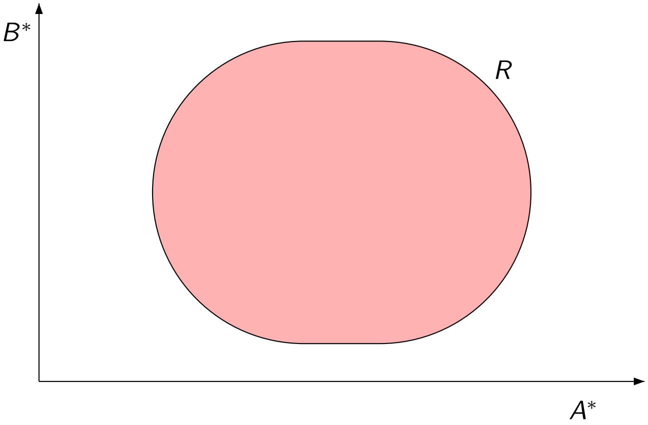
Rational bijections [Maurer, Nivat; 1980]

- The growth of a rational language is either polynomial or exponential.
- There is a rational bijection between two rational languages if and only if they have the same growth:
  - both finite with same cardinality,
  - or both polynomial with same degree for the minimal polynomial,
  - or both exponential.

A characterization of channels

A relation R has a channel iff there are two rational languages L_0 and L_1 with exponential growth such that $R \cap (L_0 \times L_1)$ is a bijection between L_0 and L_1.



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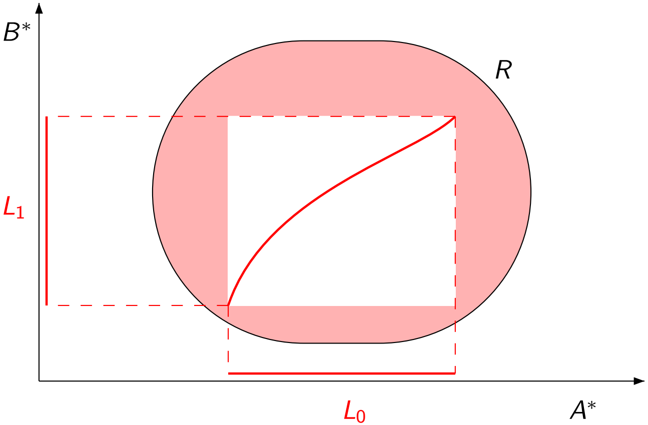

Uses the Uniformization Theorem [Schützenberger]: Any rational relation R contains a rational function with same domain.

Channels for bounded relations

A rational relation R is bounded by k iff there exist k rational functions $f_1, \ldots, f_k$ such that $R = f_1 + \cdots + f_k$. [Weber; 1996], [Sakarovitch, de Souza; 2008].

For a bounded relation R

$R = f_1 + \cdots + f_k$, where each f_i is a rational function:

- R has a channel
- iff at least one f_i has a channel
- iff *range*(R) has an exponential growth

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$R = f_1 + \cdots + f_k$, where each f_i is a rational function:

- R has a channel
- iff at least one f_i has a channel
- iff $range(R)$ has an exponential growth

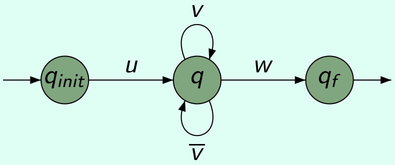
Channel synthesis

For R given as $f_1 + \cdots + f_k$, where each f_i is a rational function, the existence of a channel is decidable in linear time. When it exists, the channel can be effectively computed.

Patterns

From exponential growth to patterns

Let L be accepted by a finite automaton $\mathcal{A}$.
Then L has exponential growth if and only if $\mathcal{A}$ contains pathes:



with $v \neq \overline{v}$ and $|v| = |\overline{v}|$

This can be checked in linear time.

Definition

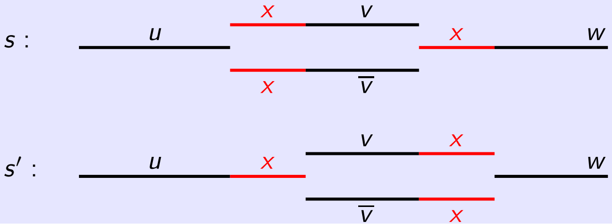
A pattern is a 4-tuple $s = (u, v, \overline{v}, w)$ such that $v \neq \overline{v}$ and $|v| = |\overline{v}|$.

- Language of s : $L_s = u(v + \overline{v})^* w$
- Subpattern of s : $s' = (ux, y, \overline{y}, zw)$, with $x, y, \overline{y}, z \in (v + \overline{v})^*$.
  Then $L_{s'} \subseteq L_s$.

Conjugated patterns

Definition

Patterns s and s' are conjugated if $s = (u, xv, x\overline{v}, xw)$ and $s' = (ux, vx, \overline{v}x, w)$ (or $s' = (ux, \overline{v}x, vx, w)$). Then $L_s = L_{s'}$.



Proposition

If s and s' are not conjugated, then one of them can be replaced by one of its subpatterns to make the associated languages disjoint.
In particular, s and s' are conjugated iff $L_s = L_{s'}$.

From patterns to channels

For patterns $s = (u, v, \overline{v}, w)$ and $s' = (u', v', \overline{v'}, w')$, the function $h_{s,s'}$ with graph $(u, u')[(v, v') + (\overline{v}, \overline{v'})]^*(w, w')$ is a rational bijection from L_s onto $L_{s'}$.

Proof of main result and channel synthesis

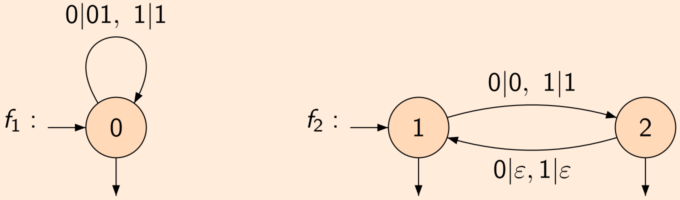
1. If f is a rational function such that $range(f)$ has exponential growth, then there exist patterns s and s' such that $h_{s,s'} \subseteq f$.
2. If R has a channel and if f is a rational function, then $R + f$ has a channel.

Procedure for $R = f_1 + \cdots + f_k$:

- Find a channel for the first f_i with exponential growth, using 1.
- Use 2. to iteratively build a channel for $f_i + f_{i+1}, \ldots, f_i + \cdots + f_k$.

Example: extraction of channel

$R = f_1 + f_2$



• For output pattern $s_1 = (\varepsilon, 011, 101, \varepsilon)$, $L_{s_1} \subseteq (01+1)^*$.
Corresponding input pattern: $s = (\varepsilon, 01, 10, \varepsilon)$, hence h_{s,s_1} gives a channel for f_1.

• In f_2, s induces output pattern $s_2 = (\varepsilon, 0, 1, \varepsilon)$, but $L_{s_1} \subseteq L_{s_2}$,
so h_{s,s_1} does not produce a channel for R.

• Extract sub-pattern $s' = (\varepsilon, 0101, 1010, \varepsilon)$ from s, as input pattern for f_2.
Corresponding output pattern: $s_3 = (\varepsilon, 00, 11, \varepsilon)$ not conjugated with s_1,
and $L_{s_1} \cap L_{s_3} = \emptyset$.

• Channel (E, D) for $R = f_1 + f_2$ is built from h_{s',s_3}:
$E(0) = 0101$, $E(1) = 1010$, and $D(00) = 0$, $D(11) = 1$.

Conclusion

Contribution

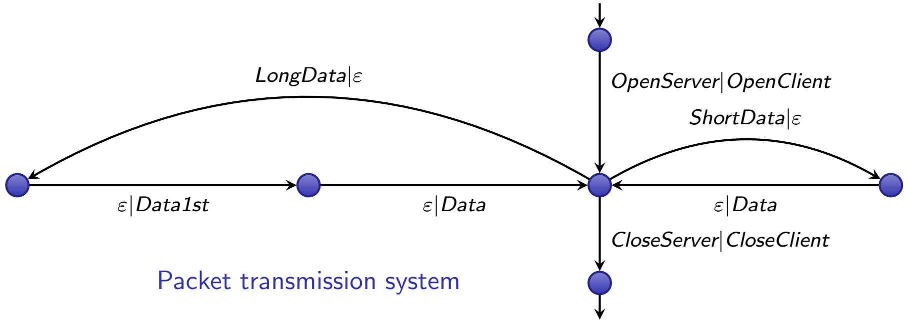
- We link the existence of a rational channel with the growth of rational languages, leading to new characterizations.
- As a consequence, we obtain a linear procedure to decide channel existence for a bounded transducer given as a sum of functions, and synthesis when the answer is positive.

Future work

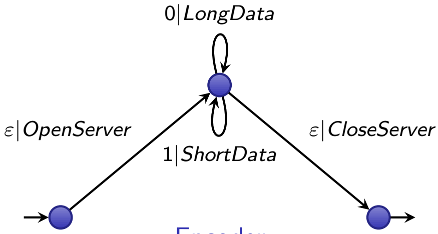
- Investigate more powerful channels, with two-ways or pushdown transducers.
- Extend the characterization to relations R such that the size of $R(u)$ is bounded by a polynomial in $|u|$.

Thank you

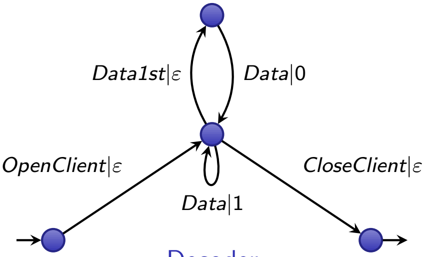
A small example of channel



Packet transmission system



Encoder



Decoder