Verification of Hybrid Systems

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Based on joint work with:
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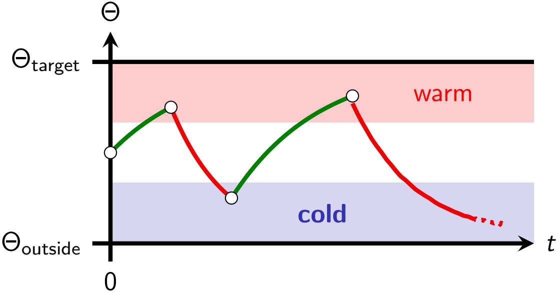
GALA, December 14th, 2019

Hybrid systems

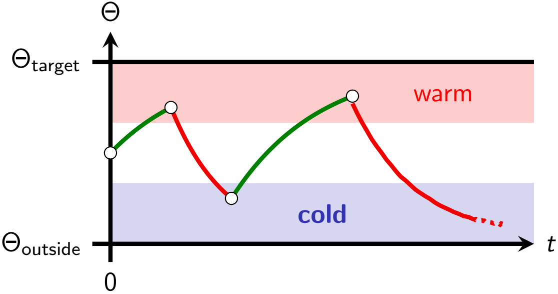
Hybrid systems



Two modes:

1. Heater **ON**: $\dot{\Theta} = \alpha(\Theta_{\text{target}} - \Theta)$
2. Heater **OFF**: $\dot{\Theta} = \beta(\Theta_{\text{outside}} - \Theta)$

Hybrid systems

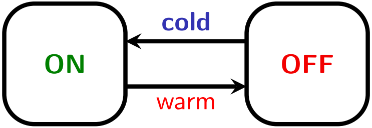


Two modes:

1. Heater **ON**: $\dot{\Theta} = \alpha(\Theta_{\text{target}} - \Theta)$
2. Heater **OFF**: $\dot{\Theta} = \beta(\Theta_{\text{outside}} - \Theta)$

Duality between:

- **Discrete** set of system modes
- **Continuous** system evolution



Thanks to V. Jugé

Verification

Verification problems are mostly undecidable on hybrid systems

Decidability requires restricting:

- either the flows [Henzinger et al. 1998]

  for instance with clocks: $\dot{x} = 1$ in all modes

- or the jumps [Alur et al. 2000]

  using for instance strong resets between modes

Other approaches

like

- bounded delay reachability,
- or approximations by discrete transition systems.

Outline

Timed Automata from Alur, Dill (1990)

Polynomial Interrupt Timed Automata

Reachability using cylindrical decomposition

Algorithmic issues

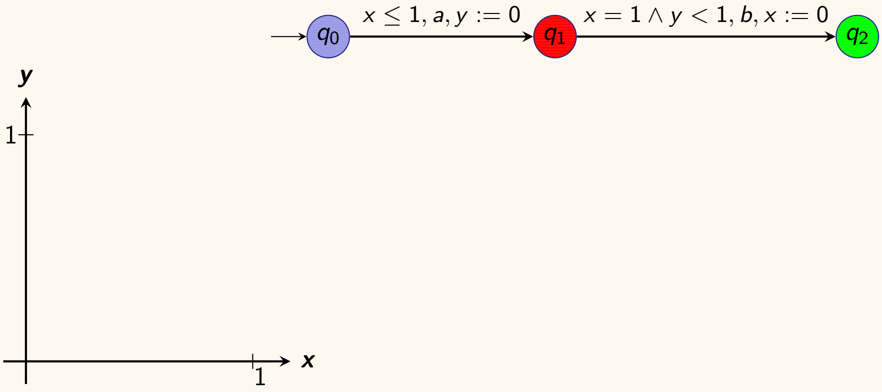
A result on Dynamical Systems

Timed automata

Variables: clocks with flow $\dot{x} = 1$ for each $x \in X$
Guards: conjunctions of $x \bowtie k$, with $k \in \mathbb{N}$ and $\bowtie$ in $\{<, \leq, =, \geq, >\}$
Updates: conjunctions of reset $x := 0$

Clock valuation: $v = (v(x_1), \ldots, v(x_n)) \in \mathbb{R}_+^n$ if $X = \{x_1, \ldots, x_n\}$

A geometric view of a trajectory



Timed automata

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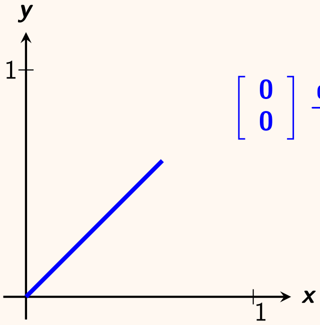
A geometric view of a trajectory

$$\to q_0 \xrightarrow{x \leq 1, a, y := 0} q_1 \xrightarrow{x = 1 \wedge y < 1, b, x := 0} q_2$$



$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \xrightarrow{0.6} \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix}$$

Timed automata

Variables: clocks with flow $\dot{x} = 1$ for each $x \in X$
Guards: conjunctions of $x \bowtie k$, with $k \in \mathbb{N}$ and $\bowtie$ in $\{<, \leq, =, \geq, >\}$
Updates: conjunctions of reset $x := 0$

Clock valuation: $v = (v(x_1), \ldots, v(x_n)) \in \mathbb{R}^n_+$ if $X = \{x_1, \ldots, x_n\}$

A geometric view of a trajectory

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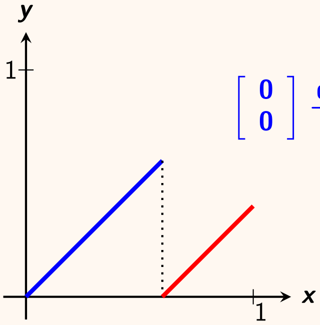

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} \xrightarrow{0.6} \begin{bmatrix} 0.6 \\ 0.6 \end{bmatrix} \xrightarrow{a} \begin{bmatrix} 0.6 \\ 0 \end{bmatrix} \xrightarrow{0.4} \begin{bmatrix} 1 \\ 0.4 \end{bmatrix}$$

Timed automata

Variables: clocks with flow $\dot{x} = 1$ for each $x \in X$
Guards: conjunctions of $x \bowtie k$, with $k \in \mathbb{N}$ and $\bowtie$ in $\{<, \leq, =, \geq, >\}$
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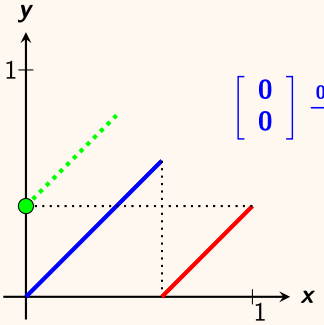


$$\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} \xrightarrow{0.6} \begin{bmatrix} \mathbf{0.6} \\ \mathbf{0.6} \end{bmatrix} \xrightarrow{a} \begin{bmatrix} \mathbf{0.6} \\ \mathbf{0} \end{bmatrix} \xrightarrow{0.4} \begin{bmatrix} \mathbf{1} \\ \mathbf{0.4} \end{bmatrix} \xrightarrow{b} \begin{bmatrix} \mathbf{0} \\ \mathbf{0.4} \end{bmatrix}$$

Reachability

Semantics of $\mathcal{A}$

with clocks $X = \{x_1, \dots, x_n\}$, set of modes Q, set of transitions E:
a transition system $\mathcal{T}_{\mathcal{A}}$ with

- **configurations:** $(q, v) \in Q \times \mathbb{R}_+^n$
- **time steps:** $(q, v) \xrightarrow{d} (q, v + d)$
- **discrete steps:** $(q, v) \xrightarrow{e} (q', v')$ for a transition $e = q \xrightarrow{g,a,r} q'$ in E if clock values v satisfy the guard g and $v' = v[r]$

An execution is a sequence alternating time and discrete steps.

Reachability problem

Given $\mathcal{A}$ and $q_f \in Q$
is there an execution from initial configuration $s_0 = (q_0, \mathbf{0})$ to (q_f, v)
for some valuation v ?

A finite quotient for timed automata

[Alur, Dill, 1990]

From $\mathcal{A}$, build a finite automaton $Reg(\mathcal{A})$ preserving reachability.

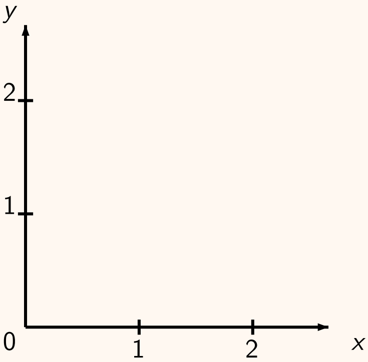
Equivalence $\sim$ over $\mathbb{R}_+^n$ producing a partition $\mathcal{R}$ of regions

The automaton $Reg(\mathcal{A})$ is time-abstract bisimilar to $\mathcal{T}_{\mathcal{A}}$:

- set of states $Q \times \mathcal{R}$,
- abstract time steps $(q, R) \to (q, succ(R))$ consistent with time elapsing in $\mathcal{T}_{\mathcal{A}}$,
- discrete steps $(q, R) \xrightarrow{e} (q', R')$ consistent with discrete transitions in $\mathcal{T}_{\mathcal{A}}$.

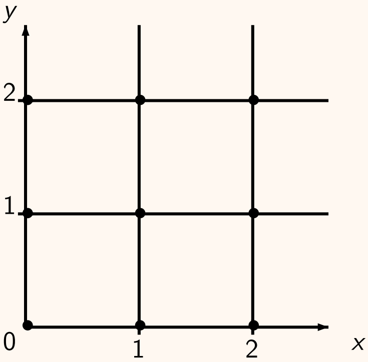
Quotient construction

A geometric view with two clocks x and y, maximal constant $m = 2$



Quotient construction

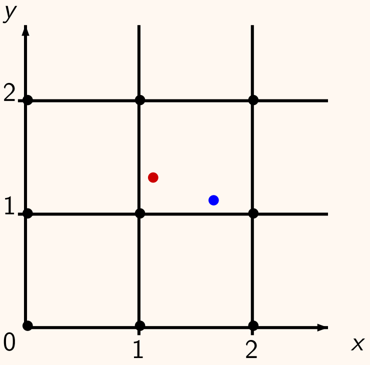
A geometric view with two clocks x and y, maximal constant $m = 2$



- Equivalent valuations must be consistent with constraints $x \bowtie k$

Quotient construction

A geometric view with two clocks x and y, maximal constant $m = 2$



- Equivalent valuations must be consistent with constraints $x \bowtie k$
- Equivalent valuations must be consistent with time elapsing

Quotient construction

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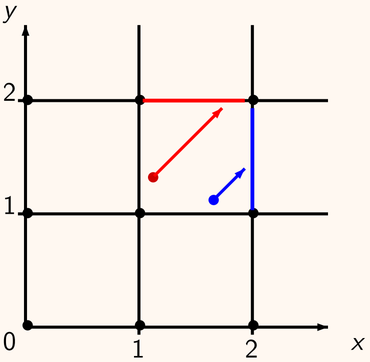

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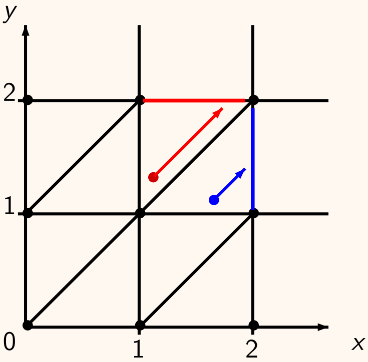

number of regions in $O(|X|! \cdot m^{|X|})$

- Equivalent valuations must be consistent with constraints $x \bowtie k$
- Equivalent valuations must be consistent with time elapsing

Quotient construction

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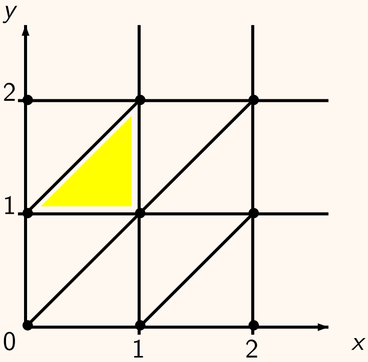
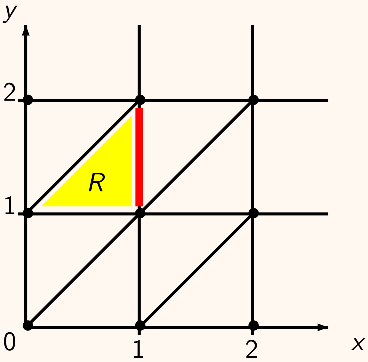


region R defined by
$0 < x < 1$ and $1 < y < 2$
and $y < x + 1$

Quotient construction

A geometric view with two clocks x and y, maximal constant $m = 2$



region R defined by
$0 < x < 1$ and $1 < y < 2$
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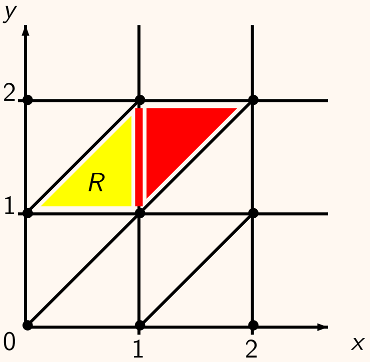
Time successor of R
$x = 1$ and $1 < y < 2$

Quotient construction

A geometric view with two clocks x and y, maximal constant $m = 2$



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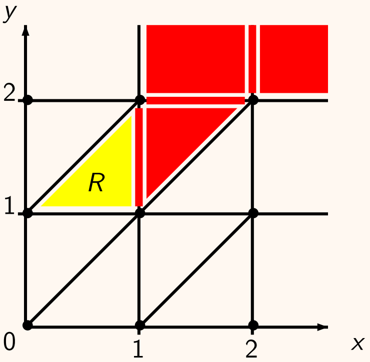

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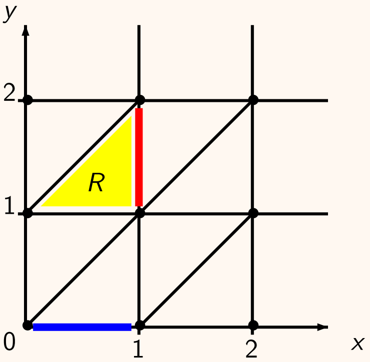
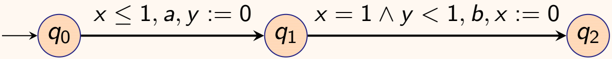


region R defined by
$0 < x < 1$ and $1 < y < 2$
and $y < x + 1$

Time successor of R
$x = 1$ and $1 < y < 2$

Discrete step from R
with $y := 0$
$0 < x < 1$ and $y = 0$

Example of quotient



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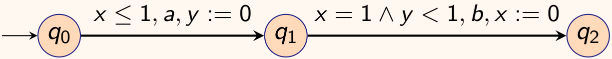

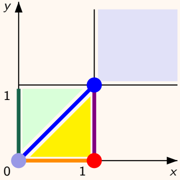

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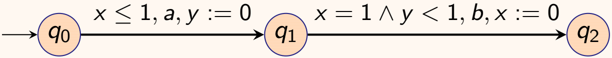

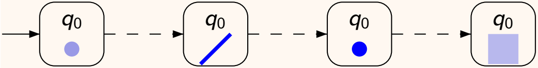

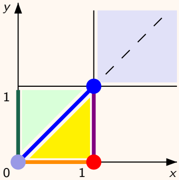

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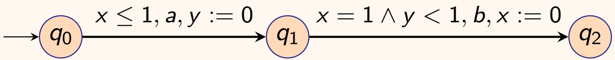

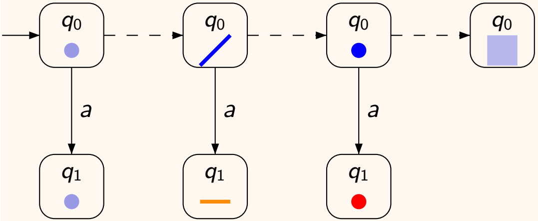

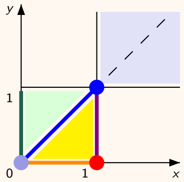

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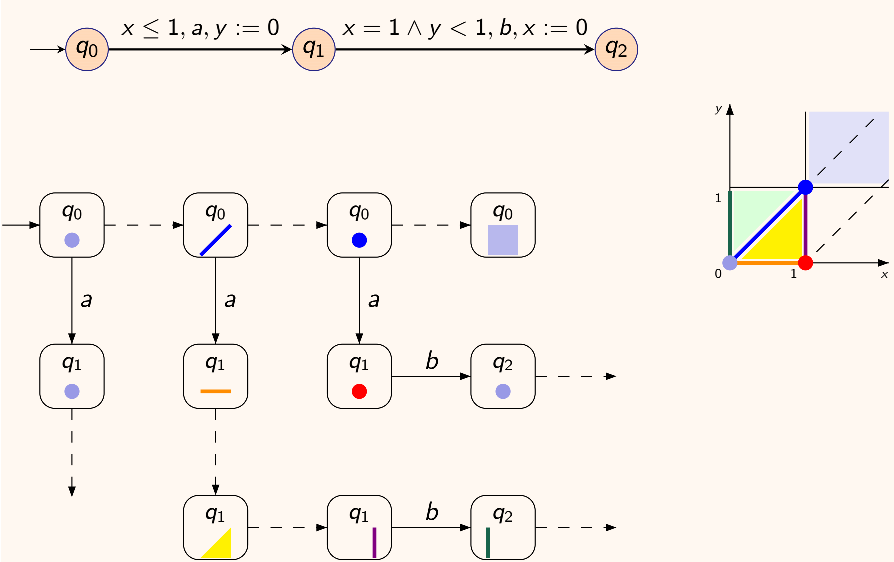


Outline

Timed Automata from Alur, Dill (1990)

Polynomial Interrupt Timed Automata

Reachability using cylindrical decomposition

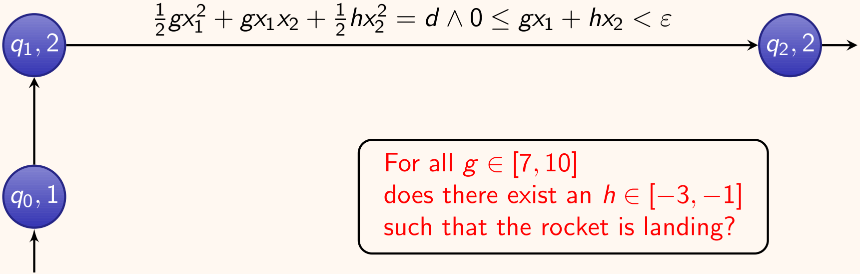
Algorithmic issues

A result on Dynamical Systems

Polynomial constraints with parameters

Landing a rocket

- First stage (lasting x_1) in state q_0:
  From distance d, the rocket approaches the land under gravitation g;
- Second stage (lasting x_2, while x_1 is frozen) in q_1:
  The rocket approaches the land with constant deceleration $h < 0$;
- Third stage: The rocket must reach the land
  with small positive speed (less than ε).

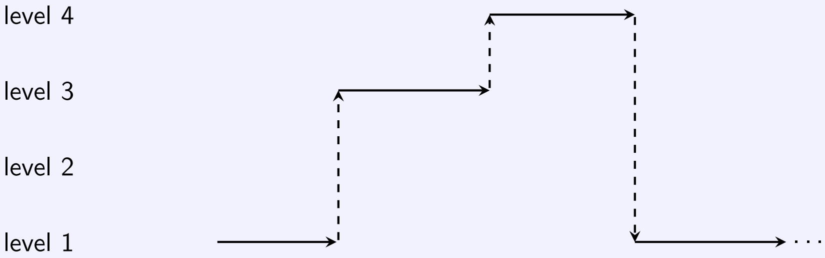


For all $g \in [7, 10]$
does there exist an $h \in [-3, -1]$
such that the rocket is landing?

Interrupt clocks

Many real-time systems include interruption mechanisms (as in processors).

Several levels with exactly one active clock at each level

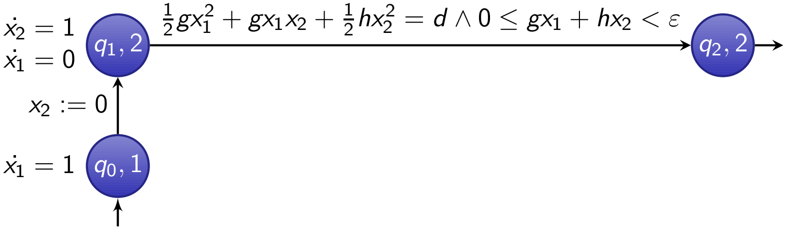


$$\begin{bmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{bmatrix} \quad \text{Exec:} \quad \begin{bmatrix} {\color{red}0} \\ {\color{red}0} \\ {\color{red}0} \\ 0 \end{bmatrix} \xrightarrow{1.5} \begin{bmatrix} {\color{red}0} \\ 0 \\ 0 \\ 1.5 \end{bmatrix} \xrightarrow{2.1} \begin{bmatrix} 0 \\ 2.1 \\ 0 \\ 1.5 \end{bmatrix} \xrightarrow{1.7} \begin{bmatrix} {\color{red}0} \\ {\color{red}0} \\ {\color{red}0} \\ 1.5 \end{bmatrix} \xrightarrow{2.2} \begin{bmatrix} {\color{red}0} \\ {\color{red}0} \\ {\color{red}0} \\ 3.7 \end{bmatrix}$$

Polynomial Interrupt Timed Automata

In the class PolITA

- variables are interrupt clocks with flow $\dot{x} = 0$ or $\dot{x} = 1$
  ordered along hierarchical levels,
- guards are polynomial constraints
  and variables can be updated by polynomials.



Main result: Reachability is decidable in 2EXPTIME

[BHPSS 15]

PolITA: Syntax

clocks $X = \{x_1, \ldots, x_n\}$ with x_k active at level k,
set of modes Q with $\lambda : Q \to \{1, \ldots, n\}$ the state level,
Guards: conjunctions of polynomial constraints
$P \bowtie 0$ with $\bowtie$ in $\{<, \leq, =, \geq, >\}$, and $P \in \mathbb{Q}[x_1, \ldots, x_k]$ at level k.

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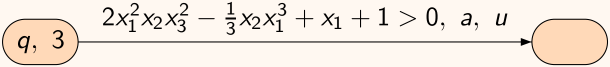

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$$
\boxed{q_1,\ 2} \xrightarrow{\quad x_2 > 2x_1^2,\ \begin{array}{l} (x_4 := 0) \\ (x_3 := 0) \\ x_2 := x_1^2 - x_1 \\ (x_1 := x_1) \end{array} \quad} \boxed{q_2,\ 4}
$$

Updates for increasing levels $k \leq k'$

Level $i > k$: reset
Level k: unchanged or polynomial update $x_k := P$ for some $P \in \mathbb{Q}[x_1, \ldots, x_{k-1}]$
Level $i < k$: unchanged.

PolITA: Syntax

clocks $X = \{x_1, \dots, x_n\}$ with x_k active at level k,
set of modes Q with $\lambda : Q \to \{1, \dots, n\}$ the state level,
Guards: conjunctions of polynomial constraints
$P \bowtie 0$ with $\bowtie$ in $\{<, \leq, =, \geq, >\}$, and $P \in \mathbb{Q}[x_1, \dots, x_k]$ at level k.

$$
\begin{tikzcd}[column sep=huge]
\boxed{q_1,\ 2} \arrow[r, "{x_2 > 2x_1^2,\ \begin{array}{l}(x_4 := 0)\\ (x_3 := 0)\\ x_2 := x_1^2 - x_1\\ (x_1 := x_1)\end{array}}"] & \boxed{q_2,\ 4} \arrow[r, "{x_4 = 3x_1^2 x_2 + x_3,\ \begin{array}{l}(x_4 := 0)\\ (x_3 := x_3)\\ (x_2 := x_2)\\ (x_1 := x_1)\end{array}}"] & \boxed{q_3,\ 3}
\end{tikzcd}
$$

Updates for decreasing levels $k > k'$

Level $i > k'$: reset
Otherwise: unchanged.

PolITA: semantics

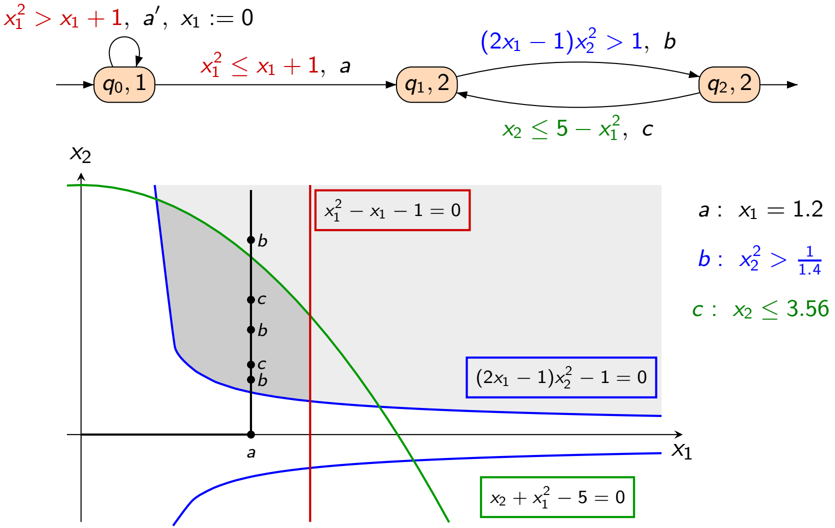
Clock valuations: $v = (v(x_1), \ldots, v(x_n)) \in \mathbb{R}^n$

The semantics of $\mathcal{A}$ is the transition system $\mathcal{T}_\mathcal{A}$

- **configurations** $S = Q \times \mathbb{R}^n$, initial configuration $s_0 = (q_0, \mathbf{0})$
- **time steps** from q at level k: $(q, v) \xrightarrow{d} (q, v +_k d)$, only x_k is active, with all clock values in $v +_k d$ unchanged except $(v +_k d)(x_k) = v(x_k) + d$
- **discrete steps** $(q, v) \xrightarrow{e} (q', v')$ for a transition $e : q \xrightarrow{g,a,u} q'$ if v satisfies the guard g and $v' = v[u]$.

An execution is a sequence alternating time and discrete steps.

Semantics: example



$(q_0, 0, 0) \xrightarrow{1.2} (q_0, 1.2, 0) \xrightarrow{a} (q_1, 1.2, 0) \xrightarrow{0.97} (q_1, 1.2, 0.97) \xrightarrow{b} (q_2, 1.2, 0.97) \ldots$

Blue and green curves meet at real roots of $-2x^5 + x_1^4 + 20x_1^3 - 10x_1^2 - 50x_1 + 26$.

Reachability problem for PolITA

Build a finite automaton $Reg(\mathcal{A})$ time-abstract bisimilar to $\mathcal{T}_\mathcal{A}$

- states: (q, C) for suitable sets of valuations $C \subseteq \mathbb{R}^n$, where polynomials of $\mathcal{A}$ have constant sign (and number of roots),
- abstract time steps: $(q, C) \to (q, succ(C))$ consistent with time elapsing in $\mathcal{T}_\mathcal{A}$,
- discrete steps: $(q, C) \xrightarrow{e} (q', C')$ consistent with discrete transitions in $\mathcal{T}_\mathcal{A}$.

The sets C will be cells from a cylindrical decomposition (CAD) adapted to the polynomials in $\mathcal{A}$.

CAD: basic example

The decomposition starts from a set of polynomials and proceeds in two phases: Elimination phase and Lifting phase.

Starting from single polynomial $P_3 = x_1^2 + x_2^2 + x_3^2 - 1 \in \mathbb{Q}[x_1, x_2][x_3]$

Elimination phase

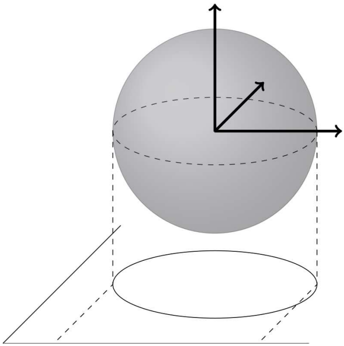
Produces polynomials in $\mathbb{Q}[x_1, x_2]$ and $\mathbb{Q}[x_1]$ required to determine the sign of P_3.

- First polynomial $P_2 = x_1^2 + x_2^2 - 1$ is produced.
  - If $P_2 > 0$ then P_3 has no real root.
  - If $P_2 = 0$ then P_3 has 0 as single root.
  - If $P_2 < 0$ then P_3 has two real roots.
- In turn the sign of $P_2 \in \mathbb{Q}[x_1][x_2]$ depends on $P_1 = x_1^2 - 1$.

Lifting phase

Produces partitions of $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}^3$ organized in a tree of cells where the signs of these polynomials (in $\{-1, 0, 1\}$) are constant.

Lifting phase



Level 1 : partition of $\mathbb{R}$ in 5 cells

$C_{-\infty} =]-\infty, -1[$, $C_{-1} = \{-1\}$, $C_0 =]-1, 1[$, $C_1 = \{1\}$, $C_{+\infty} =]1, +\infty[$

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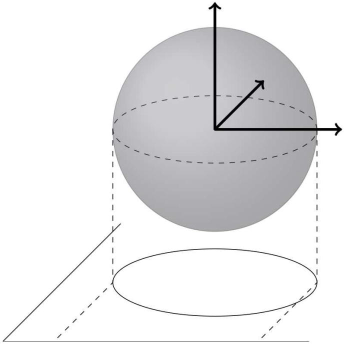

Level 2 : partition of $\mathbb{R}^2$

Above $C_{-\infty}$: a single cell $C_{-\infty} \times \mathbb{R}$

Above C_{-1}: three cells

$\{-1\}\times]-\infty, 0[$, $\{(-1, 0)\}$, $\{-1\}\times]0, +\infty[$

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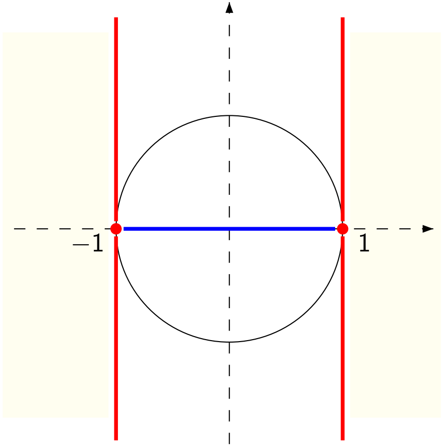

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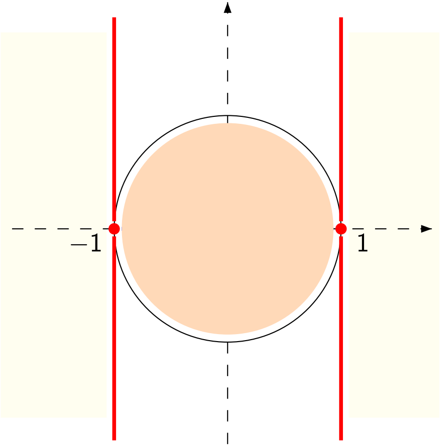

$\boxed{C_{0,0}}$ $\begin{cases} -1 < x_1 < 1 \\ -\sqrt{1-x_1^2} < x_2 < \sqrt{1-x_1^2} \end{cases}$

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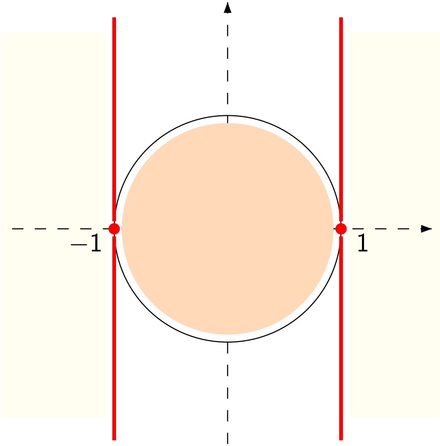

$C_{0,1}$ $\begin{cases} -1 < x_1 < 1 \\ x_2 = \sqrt{1 - x_1^2} \end{cases}$

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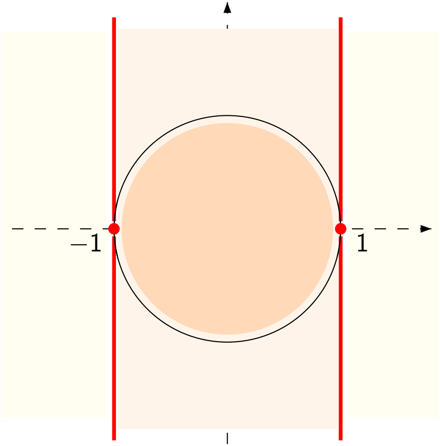

$\boxed{C_{0,+\infty}}$ $\begin{cases} -1 < x_1 < 1 \\ x_2 > \sqrt{1 - x_1^2} \end{cases}$

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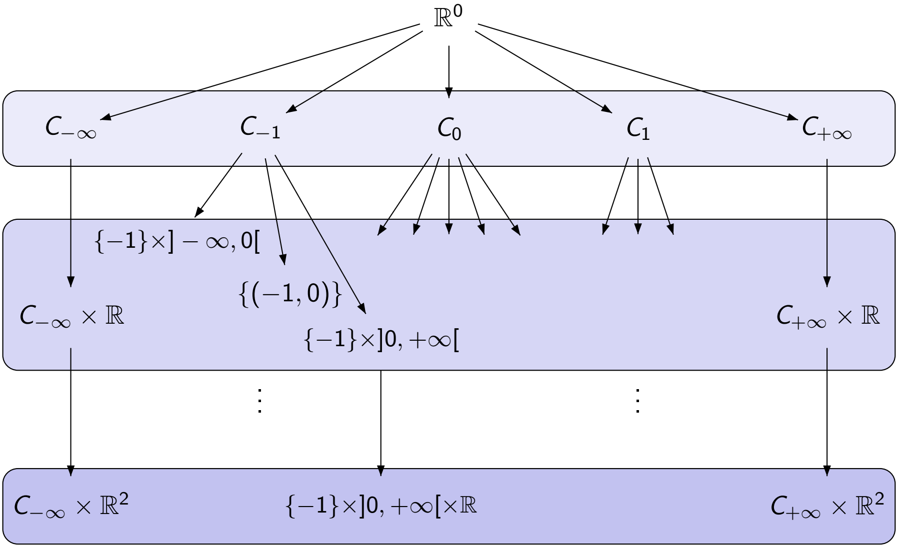
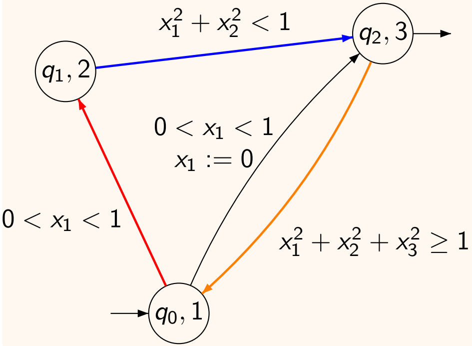
The tree of cells



Building the quotient

using the sphere case with some refinements:



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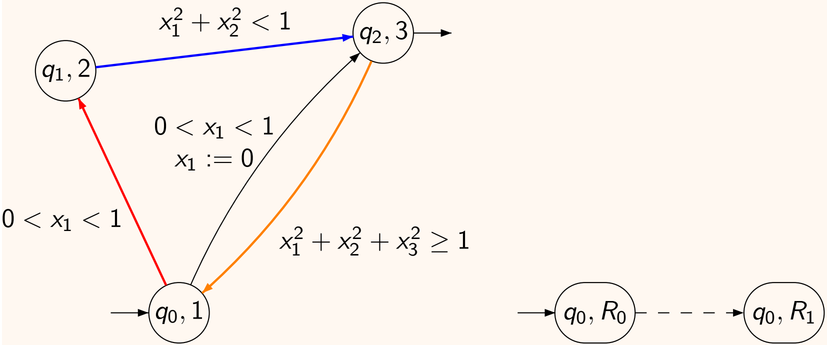

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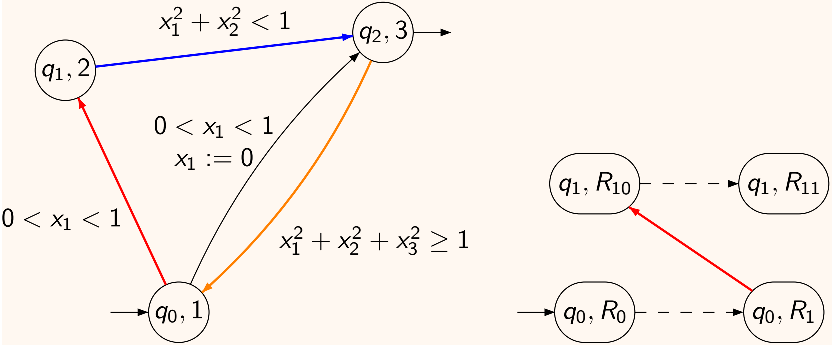

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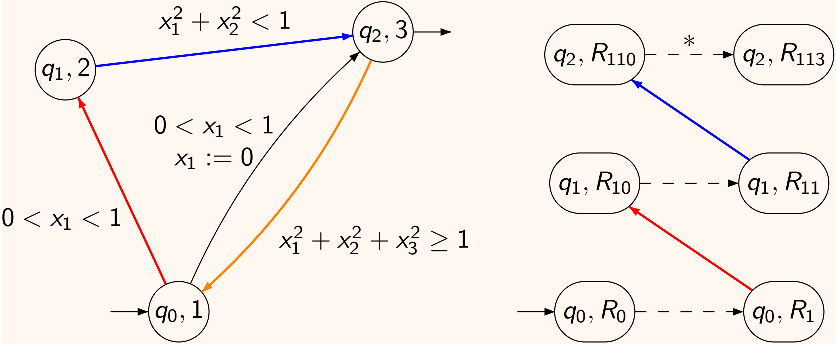

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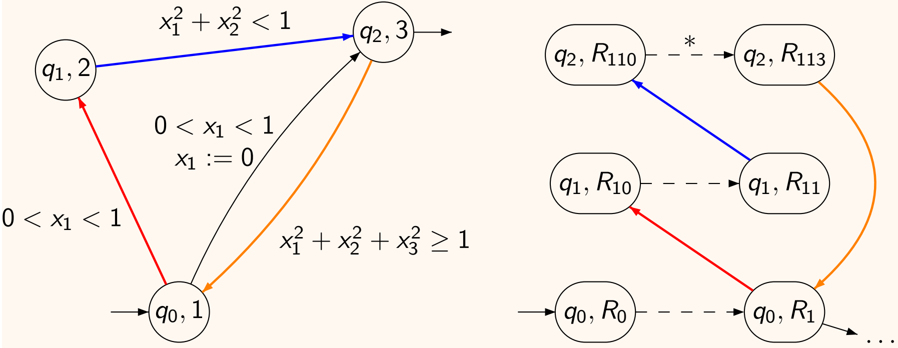

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and back to level 1

Effective construction: Elimination

From an initial set of polynomials, the elimination phase produces in 2EXPTIME a family of polynomials $\mathcal{P} = \{\mathcal{P}_k\}_{k \leq n}$ with $\mathcal{P}_k \subseteq \mathbb{Q}[x_1, \ldots, x_k]$ for level k.

Some polynomials do not always have the same degree and roots.
For instance, $B = (2x_1 - 1)x_2^2 - 1$ is of degree 2 in x_2 if and only if $x_1 \neq \frac{1}{2}$.

For $\mathcal{A}_2$

Starting from $\{x_1, A\}$ and $\{x_2, B, C\}$ with $A = x_1^2 - x_1 - 1$ and $C = x_2 + x_1^2 - 5$ results in

- $\mathcal{P}_1 = \{x_1, A, D, E, F, G\}$,
- $\mathcal{P}_2 = \{x_2, B, C\}$,

with $D = 2x_1 - 1$, $E = x_1^2 - 5$, $F = -2x_1^5 + x_1^4 + 20x_1^3 - 10x_1^2 - 50x_1 + 26$, $G = 4(2x_1 - 1)^2$

Effective construction: Lifting

To build the tree of cells in the lifting phase, we need a suitable representation of the roots of these polynomials (and the intervals between them), obtained by iteratively increasing the level.

A description like $x_3 > \sqrt{1 - x_1^2 - x_2^2}$ cannot be obtained in general.

- A point is coded by "the n^{th} root of P".
- The interval $](n, P), (m, Q)[$ is coded by a root of $(PQ)'$.

This lifting phase can be performed on-the-fly, producing only the reachable part of the quotient automaton $Reg(\mathcal{A})$.

Outline

Timed Automata from Alur, Dill (1990)

Polynomial Interrupt Timed Automata

- Reachability using cylindrical decomposition
- Algorithmic issues

A result on Dynamical Systems

Dynamical systems

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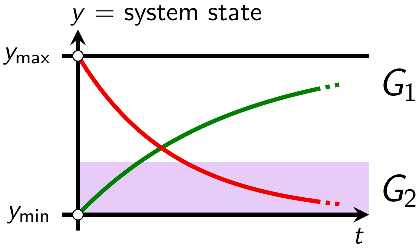

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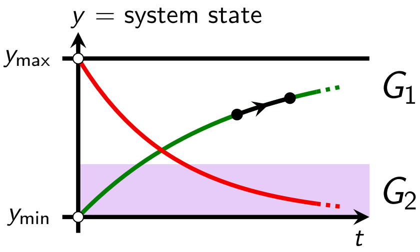

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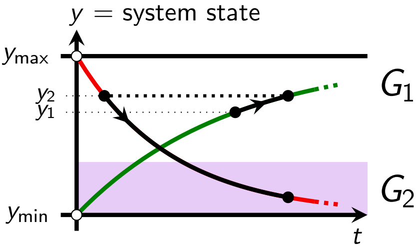

$$y_1 = f(t_1) \rightarrow y_2 = f(t_2) = g(t_3) \rightarrow y_3 = g(t_4)$$

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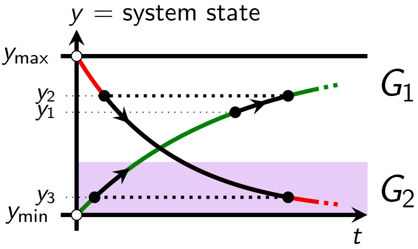

Transition system:

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Notations and examples

A dynamical system $(\mathcal{M}, \gamma)$:

- $\mathcal{M} = \langle M, \leqslant, ... \rangle$ a linearly ordered structure,
- $\gamma : V_1 \times V \to V_2$ for $V_1 \subseteq M^{k_1}$, $V \subseteq M$, $V_2 \subseteq M^{k_2}$, all (FO-)definable in $\mathcal{M}$,

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Clocks have dynamics $\gamma : \mathbb{R}_+^n \times [0, +\infty[\to \mathbb{R}_+^n$ with $\gamma(v, t) = \gamma_v(t) = v + t$.

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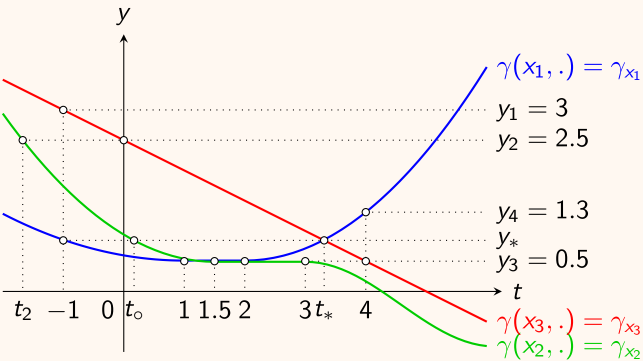

Bisimulations for dynamical systems

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Theorem [Lafferriere, Pappas, Sastry 2000]

Bisimulation is decidable and induces a finite partition when:
$\gamma : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is solution of $\mathrm{d}\gamma(x,t)/\mathrm{d}t = F(\gamma(x,t))$ definable in an o-minimal theory of $\mathbb{R}$.

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. . . and counter-examples:

- $(\mathbb{Q}, \leqslant, +, \times)$ — $x^2 \leqslant 1 + 1 \Leftrightarrow -\sqrt{2} \leqslant x \leqslant \sqrt{2}$
- $(\mathbb{Z}_{\geqslant 0}, \leqslant, +)$ — $\exists z, x = z + z \Leftrightarrow x$ is even
- $(\mathbb{R}, \leqslant, \sin)$ — $\sin(x) = 0 \Leftrightarrow x \in \pi\mathbb{Z}$

Properties

[Pillay, Steinhorn 88]

Property 1

Let $(M, \leqslant, \ldots)$ be o-minimal and $f : M \to M$ be definable. There exists a finite partition $(\mathcal{I}_1, \ldots, \mathcal{I}_k)$ of M into intervals s.t., for all $j \leqslant k$:

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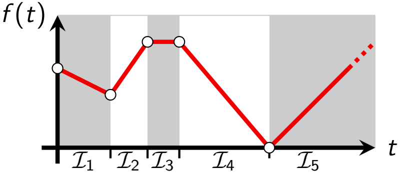

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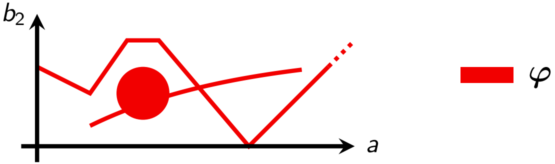
Let φ be an ℓ-variable formula. There exists $\mathbf{N}_\varphi$ s.t., for all $b_2, \ldots, b_\ell \in M$, the set $\{a \in M \mid (a, b_2, \ldots, b_\ell) \models \varphi\}$ is a union of at most $\mathbf{N}_\varphi$ intervals.



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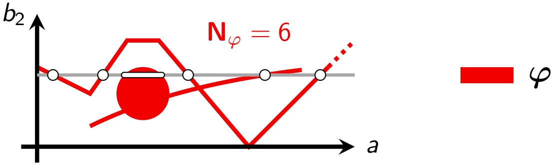

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[BBJ 18]

Generalising Lafferriere et al.:

- o-minimal **real** theory $\rightarrow$ **any** o-minimal theory
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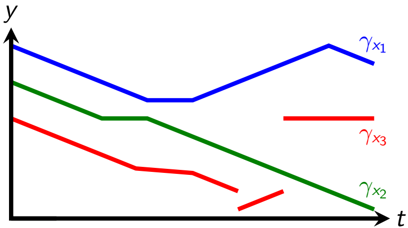

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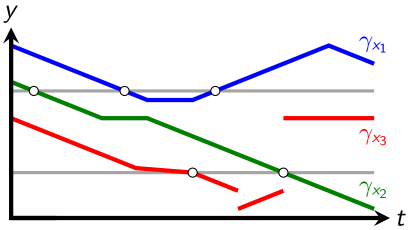

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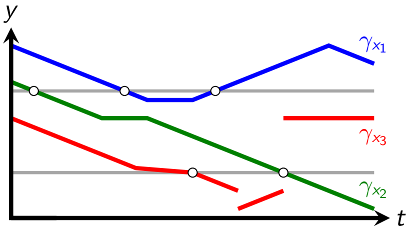

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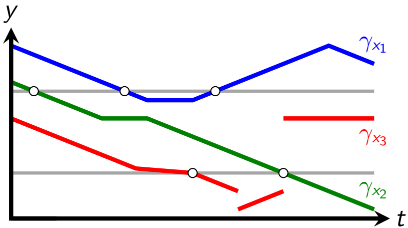

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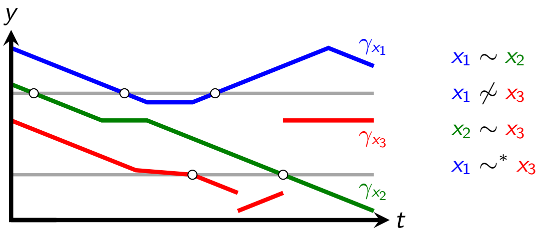
  the bisimulation relation is **definable** and induces **finite** partition.



Idea of the proof

First step: decomposition

For all $x \in V_1$ with dynamics γ_x:

- Produce a classification of time intervals into x-static or x-adaptable intervals.
- If $V_1(x) = \{x' \in V_1 \mid x \sim x'\}$ is finite, then there is a finite definable partition of the time set v into maximal x-static and x-adaptable intervals.
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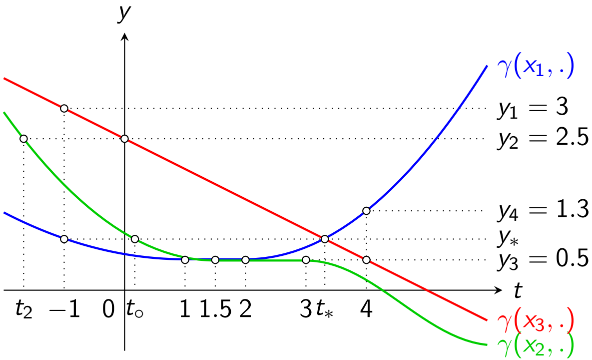
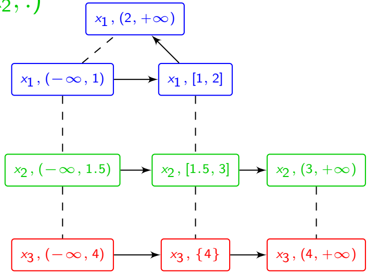
Second step: building a bisimulation graph

- with nodes $(x, \mathcal{I})$ for the intervals above,
- edges $(x, \mathcal{I}) \to (x, \mathcal{J})$ that represent time elapsing on γ_x,
- ε-edges $(x, \mathcal{I}) \to (x', \mathcal{I}')$ that represent jumps between trajectories.

Example



y
t
$\gamma(x_1, .)$
$y_1 = 3$
$y_2 = 2.5$
$y_4 = 1.3$
y_*
$y_3 = 0.5$
t_2 -1 0 $t_\circ$ 1 1.5 2 3 t_* 4
$\gamma(x_3, .)$
$\gamma(x_2, .)$



$x_1, (2, +\infty)$
$x_1, (-\infty, 1)$
$x_1, [1, 2]$
$x_2, (-\infty, 1.5)$
$x_2, [1.5, 3]$
$x_2, (3, +\infty)$
$x_3, (-\infty, 4)$
$x_3, \{4\}$
$x_3, (4, +\infty)$

Conclusion

Summary

- Reachability is decidable in two models without strong resets: Timed Automata and Polynomial Interrupt Timed Automata.
- Bisimulation is decidable in a richer model of dynamical systems, which can immediately be extended with modes and strong resets.

Going further

- Refine the crossing conditions,
- Add modes with weaker jump conditions.

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Thank you