New Applications of Moment-SOS Hierarchies

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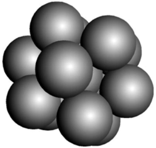
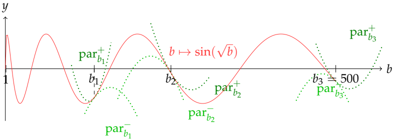
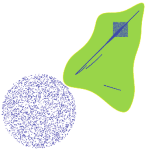
12 February 2015

Pecan Seminar
LIP6







Personal Background

- 2008 – 2010: Master at Tokyo University
  HIERARCHICAL DOMAIN DECOMPOSITION METHODS
  (S. Yoshimura)

- 2010 – 2013: PhD at Inria Saclay LIX/CMAP
  FORMAL PROOFS FOR NONLINEAR OPTIMIZATION
  (S. Gaubert and B. Werner)

- 2014 Jan-Sept: Postdoc at LAAS-CNRS
  MOMENT-SOS APPLICATIONS
  (D. Henrion and J.B. Lasserre)

- From 2014 Oct: Postdoc at Imperial
  SDP FOR AUTOMATED HARDWARE TUNING
  (G. Constantinides and A. Donaldson)

Errors and Proofs

- Mathematicians want to eliminate all the uncertainties on their results. Why?

  M. Lecat, Erreurs des Mathématiciens des origines à nos jours, 1935.

  130 pages of errors! (Euler, Fermat, Sylvester, ...)

Errors and Proofs

- Possible workaround: proof assistants

  COQ (Coquand, Huet 1984)

  HOL-LIGHT (Harrison, Gordon 1980)

  Built in top of OCAML

Computer Science and Mathematics

- Tool: Formal Bounds for Global Optimization

- Collaboration with:

  Benjamin Werner (LIX Polytechnique)

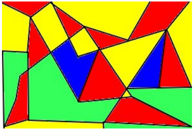
  Stéphane Gaubert (Maxplus Team CMAP/INRIA Polytechnique)

  Xavier Allamigeon (Maxplus Team)

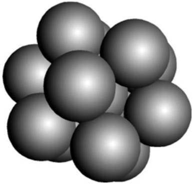
Complex Proofs

- Complex mathematical proofs / mandatory computation

K. Appel and W. Haken , Every Planar Map is Four-Colorable, 1989.



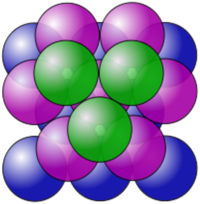
T. Hales, A Proof of the Kepler Conjecture, 1994.



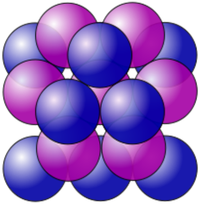
From Oranges Stack...

Kepler Conjecture (1611):

The maximal density of sphere packings in 3D-space is $\frac{\pi}{\sqrt{18}}$



Face-centered cubic Packing



Hexagonal Compact Packing

...to Flyspeck Nonlinear Inequalities

- The proof of T. Hales (1998) contains mathematical and computational parts
- Computation: check thousands of nonlinear inequalities
- Robert MacPherson, editor of The Annals of Mathematics: "[...] the mathematical community will have to get used to this state of affairs."
- **F**ly**s**pec**k** [Hales 06]: **F**ormal **P**roof of **K**epler Conjecture

...to Flyspeck Nonlinear Inequalities

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- **Flyspeck** [Hales 06]: **F**ormal **P**roof of **K**epler Conjecture

- **Project Completion on 10 August by the Flyspeck team!!**

...to Flyspeck Nonlinear Inequalities

- Nonlinear inequalities: quantified reasoning with "$\forall$"

$$\forall \mathbf{x} \in \mathbf{K}, f(\mathbf{x}) \geqslant 0$$

- NP-hard optimization problem

A “Simple” Example

In the computational part:

- Multivariate Polynomials:
  $\Delta\mathbf{x} := x_1x_4(-x_1 + x_2 + x_3 - x_4 + x_5 + x_6) + x_2x_5(x_1 - x_2 + x_3 + x_4 - x_5 + x_6) + x_3x_6(x_1 + x_2 - x_3 + x_4 + x_5 - x_6) - x_2(x_3x_4 + x_1x_6) - x_5(x_1x_3 + x_4x_6)$

A "Simple" Example

In the computational part:

- Semialgebraic functions: composition of polynomials with $|\cdot|, \sqrt{}, +, -, \times, /, \sup, \inf, \dots$

$$p(\mathbf{x}) := \partial_4 \Delta \mathbf{x} \qquad q(\mathbf{x}) := 4 x_1 \Delta \mathbf{x}$$
$$r(\mathbf{x}) := p(\mathbf{x}) / \sqrt{q(\mathbf{x})}$$

$$l(\mathbf{x}) := -\frac{\pi}{2} + 1.6294 - 0.2213\,(\sqrt{x_2} + \sqrt{x_3} + \sqrt{x_5} + \sqrt{x_6} - 8.0) + 0.913\,(\sqrt{x_4} - 2.52) + 0.728\,(\sqrt{x_1} - 2.0)$$

A “Simple” Example

In the computational part:

- Transcendental functions $\mathcal{T}$: composition of semialgebraic functions with arctan, exp, sin, $+, -, \times, \ldots$

A “Simple” Example

In the computational part:

- Feasible set $\mathbf{K} := [4, 6.3504]^3 \times [6.3504, 8] \times [4, 6.3504]^2$

Lemma$_{9922699028}$ from Flyspeck:

$$\forall \mathbf{x} \in \mathbf{K}, \arctan\left(\frac{p(\mathbf{x})}{\sqrt{q(\mathbf{x})}}\right) + l(\mathbf{x}) \geqslant 0$$

Existing Formal Frameworks

Formal proofs for Global Optimization:

- Bernstein polynomial methods [Zumkeller's PhD 08]
- SMT methods [Gao et al. 12]
- Interval analysis and Sums of squares

Existing Formal Frameworks

Interval analysis

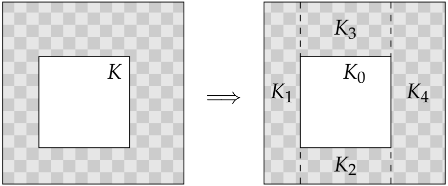
- Certified interval arithmetic in COQ [Melquiond 12]
- Taylor methods in HOL Light [Solovyev thesis 13]
  - Formal verification of floating-point operations
- robust but subject to the **Curse of Dimensionality**

Existing Formal Frameworks

Lemma$_{9922699028}$ from Flyspeck:

$$\forall \mathbf{x} \in \mathbf{K}, \arctan\left(\frac{\partial_4 \Delta \mathbf{x}}{\sqrt{4x_1 \Delta \mathbf{x}}}\right) + l(\mathbf{x}) \geqslant 0$$

- Dependency issue using Interval Calculus:
  - One can bound $\partial_4 \Delta \mathbf{x} / \sqrt{4x_1 \Delta \mathbf{x}}$ and $l(\mathbf{x})$ separately
  - Too coarse lower bound: -0.87
  - Subdivide $\mathbf{K}$ to prove the inequality



Existing Formal Frameworks

Sums of squares (SOS) techniques

- Formalized in HOL-LIGHT [Harrison 07] COQ [Besson 07]
- Precise methods but scalability and robustness issues (numerical)
- powerful: global optimality certificates without branching

but

- not so robust: handles moderate size problems
- Restricted to polynomials

Existing Formal Frameworks

- *Caprasse* Problem:

  $\forall \mathbf{x} \in [-0.5, 0.5]^4, -x_1 x_3^3 + 4 x_2 x_3^2 x_4 + 4 x_1 x_3 x_4^2 + 2 x_2 x_4^3 + 4 x_1 x_3 + 4 x_3^2 - 10 x_2 x_4 - 10 x_4^2 + 5.1801 \geqslant 0.$

- Decompose the polynomial as SOS of degree at most 4

- Gives a nonnegative bound!

Existing Formal Frameworks

The "No Free Lunch" Rule:

- Exponential dependency in
  1. Relaxation order k (SOS degree)
  2. number of variables n of the polynomial
- Computing k-th bound involves $\binom{n+2k}{n}$ variables
- At fixed k, $O(n^{2k})$ variables

Existing Formal Frameworks

Approximation theory: Chebyshev/Taylor models

- mandatory for non-polynomial problems
- hard to combine with SOS techniques (degree of approximation)

Existing Formal Frameworks

Can we develop a new approach with both keeping the respective strength of interval and precision of SOS?

Proving Flyspeck Inequalities is challenging: medium-size and tight

New Framework (in my PhD thesis)

- Certificates for lower bounds of Nonlinear optimization using:
  - Moment-SOS hierarchies
  - Maxplus approximation (Optimal Control)
- Verification of these certificates inside Coq

New Framework (in my PhD thesis)

Software Implementation `NLCertify`:

- `https://forge.ocamlcore.org/projects/nl-certify/`

15 000 lines of OCaml code

4000 lines of Coq code

Moment-SOS Hierarchies for Polynomial Optimization

New Applications of Moment-SOS Hierarchies

Conclusion

Polynomial Optimization

- Semialgebraic set $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \dots, g_l(\mathbf{x}) \geqslant 0\}$
- $p^* := \inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x})$: NP hard
- Sums of squares $\Sigma[\mathbf{x}]$
  e.g. $x_1^2 - 2x_1x_2 + x_2^2 = (x_1 - x_2)^2$
- $\mathcal{Q}(\mathbf{S}) := \left\{ \sigma_0(\mathbf{x}) + \sum_{j=1}^{l} \sigma_j(\mathbf{x}) g_j(\mathbf{x}), \text{ with } \sigma_j \in \Sigma[\mathbf{x}] \right\}$
- **REMEMBER**: $f \in \mathcal{Q}(\mathbf{S}) \implies \forall \mathbf{x} \in \mathbf{S}, f(\mathbf{x}) \geqslant 0$

Problem reformulation [Lasserre 01]

- Borel σ-algebra $\mathcal{B}$ (generated by the open sets of $\mathbb{R}^n$)
- $\mathcal{M}_+(\mathbf{S})$: set of probability measures supported on $\mathbf{S}$.
  If $\mu \in \mathcal{M}_+(\mathbf{S})$ then
  1. $\mu : \mathcal{B} \to [0, 1]$, $\mu(\emptyset) = 0$
  2. $\mu(\bigcup_i B_i) = \sum_i \mu(B_i)$, for any countable $(B_i) \subset \mathcal{B}$
  3. $\int_{\mathbf{S}} \mu(d\mathbf{x}) = 1$
- $\text{supp}(\mu)$ is the smallest set $\mathbf{S}$ such that $\mu(\mathbb{R}^n \backslash \mathbf{S}) = 0$

Problem reformulation [Lasserre 01]

$$p^* = \inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{S})} \int_{\mathbf{S}} f \, \mu(d\mathbf{x})$$

Problem reformulation [Lasserre 01]

$$p^* = \inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x}) = \inf_{\mu \in \mathcal{M}_+(\mathbf{S})} \sum_{\alpha} f_\alpha \int_{\mathbf{S}} \mathbf{x}^\alpha \, \mu(d\mathbf{x})$$

Primal-dual Moment-SOS [Lasserre 01]

- Let $(\mathbf{x}^\alpha)_{\alpha \in \mathbb{N}^n}$ be the monomial basis

Definition

A sequence $\mathbf{z}$ has a representing measure on $\mathbf{S}$ if there exists a finite measure μ supported on $\mathbf{S}$ such that

$$\mathbf{z}_\alpha = \int_{\mathbf{S}} \mathbf{x}^\alpha \mu(d\mathbf{x}), \quad \forall \alpha \in \mathbb{N}^n.$$

Primal-dual Moment-SOS [Lasserre 01]

- $\mathcal{M}_+(\mathbf{S})$: space of probability measures supported on $\mathbf{S}$
- $\mathcal{Q}(\mathbf{S})$: quadratic module

Polynomial Optimization Problems (POP)

$$
\begin{array}{llcll}
 & \text{(Primal)} & & & \text{(Dual)} \\
\inf & \int_{\mathbf{S}} f \, d\mu & = & \sup & \lambda \\
\text{s.t.} & \mu \in \mathcal{M}_+(\mathbf{S}) & & \text{s.t.} & \lambda \in \mathbb{R} \,, \\
 & & & & f - \lambda \in \mathcal{Q}(\mathbf{S})
\end{array}
$$

Primal-dual Moment-SOS [Lasserre 01]

- Finite moment sequences $\mathbf{z}$ of measures in $\mathcal{M}_+(\mathbf{S})$
- Truncated quadratic module $\mathcal{Q}_k(\mathbf{S}) := \mathcal{Q}(\mathbf{S}) \cap \mathbb{R}_{2k}[\mathbf{x}]$

Polynomial Optimization Problems (POP)

$$
\begin{array}{llcll}
 & \text{(Moment)} & & & \text{(SOS)} \\
\inf & \sum_{\alpha} f_\alpha \, \mathbf{z}_\alpha & = & \sup & \lambda \\
\text{s.t.} & \mathbf{M}_{k-v_j}(g_j \, \mathbf{z}) \succcurlyeq 0 \,, \quad 0 \leqslant j \leqslant l, & & \text{s.t.} & \lambda \in \mathbb{R} \;, \\
 & \mathbf{z}_1 = 1 & & & f - \lambda \in \mathcal{Q}_k(\mathbf{S})
\end{array}
$$

Semidefinite Optimization

- F_0, F_α symmetric real matrices, cost vector c

Primal-dual pair of semidefinite programs:

$$(SDP) \begin{cases} \mathcal{P}: & \inf_{\mathbf{z}} & \sum_\alpha c_\alpha \mathbf{z}_\alpha \\ & \text{s.t.} & \sum_\alpha F_\alpha \, \mathbf{z}_\alpha - F_0 \succcurlyeq 0 \\ \\ \mathcal{D}: & \sup_{\mathbf{Y}} & \operatorname{Trace}(F_0 \, \mathbf{Y}) \\ & \text{s.t.} & \operatorname{Trace}(F_\alpha \, \mathbf{Y}) = c_\alpha \ , \quad \mathbf{Y} \succcurlyeq 0 \ . \end{cases}$$

- Freely available SDP solvers (CSDP, SDPA, SEDUMI)

Moment-SOS Hierarchies for Polynomial Optimization

New Applications of Moment-SOS Hierarchies

Conclusion

General informal Framework

Given $\mathbf{K}$ a compact set and f a transcendental function, bound
$f^* = \inf_{\mathbf{x} \in \mathbf{K}} f(\mathbf{x})$ and prove $f^* \geqslant 0$

- f is under-approximated by a semialgebraic function f_{sa}
- Reduce the problem $f_{\text{sa}}^* := \inf_{\mathbf{x} \in \mathbf{K}} f_{\text{sa}}(\mathbf{x})$ to a polynomial optimization problem (POP)

General informal Framework

Approximation theory: Chebyshev/Taylor models

- mandatory for non-polynomial problems
- hard to combine with Sum-of-Squares techniques (degree of approximation)

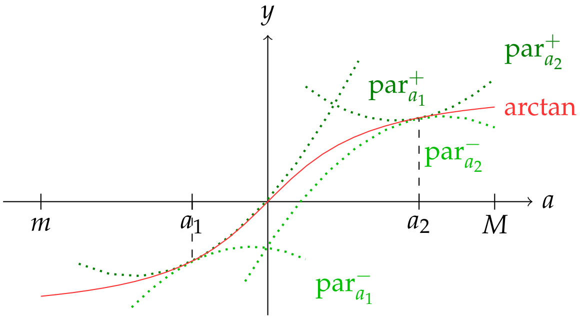
Maxplus Approximation

- Initially introduced to solve Optimal Control Problems [Fleming-McEneaney 00]
- **Curse of dimensionality** reduction [McEaneney Kluberg, Gaubert-McEneaney-Qu 11, Qu 13].
  Allowed to solve instances of dim up to 15 (inaccessible by grid methods)
- In our context: approximate transcendental functions

Maxplus Approximation

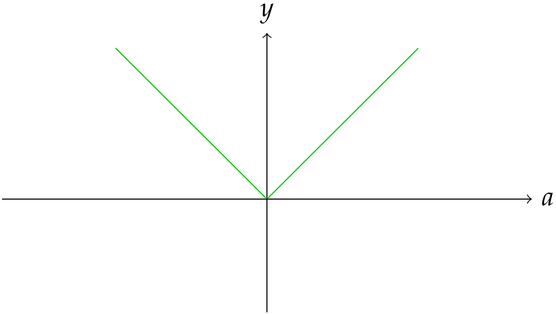
Definition

Let $\gamma \geqslant 0$. A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is said to be γ-*semiconvex* if the function $\mathbf{x} \mapsto \phi(\mathbf{x}) + \frac{\gamma}{2} \|\mathbf{x}\|_2^2$ is convex.



Nonlinear Function Representation

Exact parsimonious maxplus representations



Nonlinear Function Representation

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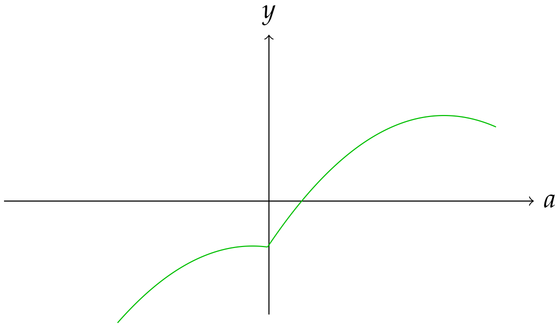


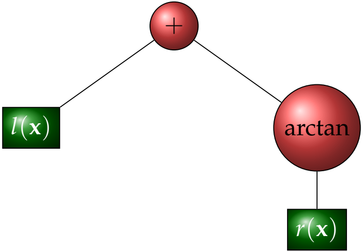
Nonlinear Function Representation

Abstract syntax tree representations of multivariate transcendental functions:

- leaves are semialgebraic functions of $\mathcal{A}$
- nodes are univariate functions of $\mathcal{D}$ or binary operations

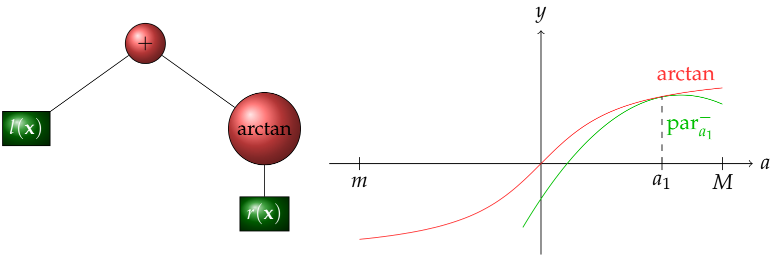
Nonlinear Function Representation

- For the “Simple” Example from Flyspeck:



Maxplus Optimization Algorithm

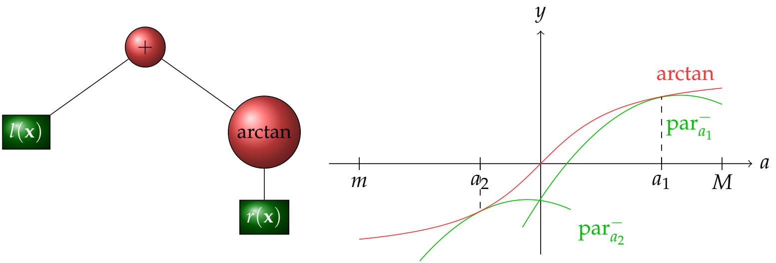
First iteration:



1. control point $\{a_1\}$: $m_1 = -4.7 \times 10^{-3} < 0$

Maxplus Optimization Algorithm

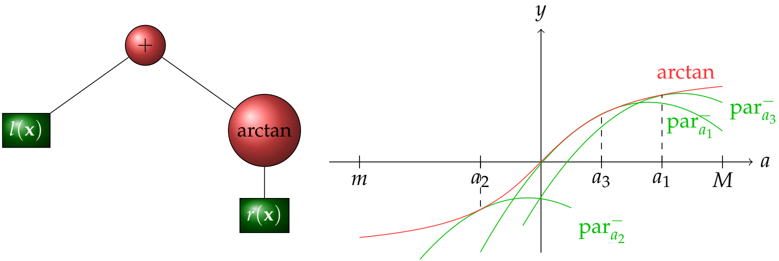
Second iteration:



2 control points $\{a_1, a_2\}$: $m_2 = -6.1 \times 10^{-5} < 0$

Maxplus Optimization Algorithm

Third iteration:



3 control points $\{a_1, a_2, a_3\}$: $m_3 = 4.1 \times 10^{-6} > 0$

OK!

Contributions

V. Magron, X. Allamigeon, S. Gaubert, and B. Werner. Certification of Real Inequalities – Templates and Sums of Squares, arxiv:1403.5899, 2014. Accepted for publication in *Mathematical Programming SERIES B, volume on Polynomial Optimization*.

The General "Formal Framework"

- We check the correctness of SOS certificates for POP
- We build certificates to prove interval bounds for semialgebraic functions
- We bound formally transcendental functions with semialgebraic approximations

Formal SOS bounds

When $q \in \mathcal{Q}(\mathbf{K}), \sigma_0, \dots, \sigma_m$ is a positivity certificate for q
Check **symbolic polynomial equalities** $\boxed{q = q'}$ in COQ

- Existing tactic `ring` [Grégoire-Mahboubi 05]

- Polynomials coefficients: arbitrary-size rationals `bigQ` [Grégoire-Théry 06]

- Much simpler to verify certificates using *sceptical approach*

- Extends also to semialgebraic functions

Formal Nonlinear Optimization

Inequality	#boxes	Time	Time
9922699028	39	190 s	2218 s
3318775219	338	1560 s	19136 s

- Comparable with Taylor interval methods in HOL-LIGHT [Hales-Solovyev 13]

Bottleneck of informal optimizer is SOS solver

22 times slower! $\Longrightarrow$ Current bottleneck is to check polynomial equalities

Contribution

For more details on the formal side:

V. M., X. Allamigeon, S. Gaubert and B. Werner.
Formal Proofs for Nonlinear Optimization,
arxiv:1404.7282, 2015. *Journal of Formalized Reasoning*.

Bicriteria Optimization Problems

- Let $f_1, f_2 \in \mathbb{R}[\mathbf{x}]$ two conflicting criteria
- Let $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_m(\mathbf{x}) \geqslant 0\}$ a semialgebraic set

$$(\mathbf{P}) \left\{ \min_{\mathbf{x} \in \mathbf{S}} (f_1(\mathbf{x})\, f_2(\mathbf{x}))^\top \right\}$$

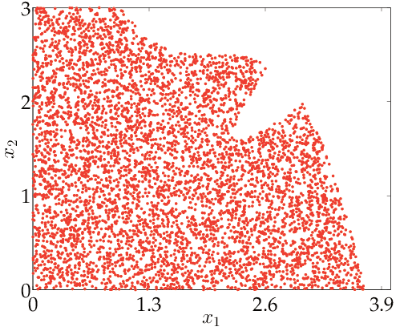
Assumption

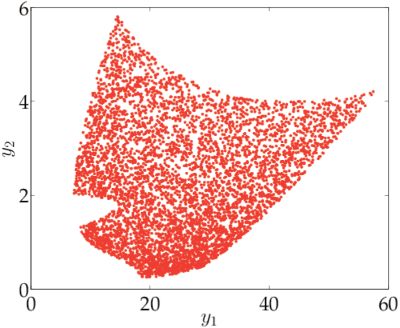
The image space $\mathbb{R}^2$ is partially ordered in a natural way ($\mathbb{R}^2_+$ is the ordering cone).

Bicriteria Optimization Problems

$$g_1 := -(x_1 - 2)^3/2 - x_2 + 2.5 \ ,$$
$$g_2 := -x_1 - x_2 + 8(-x_1 + x_2 + 0.65)^2 + 3.85 \ ,$$
$$\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : g_1(\mathbf{x}) \geqslant 0, g_2(\mathbf{x}) \geqslant 0\} \ .$$

$$f_1 := (x_1 + x_2 - 7.5)^2/4 + (-x_1 + x_2 + 3)^2 \ ,$$
$$f_2 := (x_1 - 1)^2/4 + (x_2 - 4)^2/4 \ .$$





Parametric Sublevel Set Approximations

- Inspired by previous research on multiobjective linear optimization [Gorissen-den Hertog 12]

- Workaround: reduce $\mathbf{P}$ to a **parametric POP**

$$(\mathbf{P}_\lambda): \quad f^*(\lambda) := \min_{\mathbf{x} \in \mathbf{S}} \{ f_2(\mathbf{x}) : f_1(\mathbf{x}) \leqslant \lambda \} \ ,$$

- variable $(\mathbf{x}, \lambda) \in \mathbf{K} = \mathbf{S} \times [0, 1]$

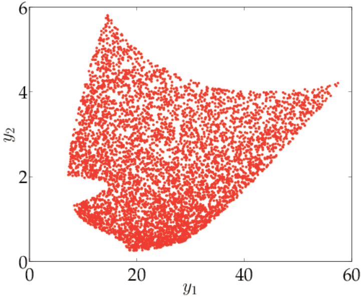
A Hierarchy of Polynomial Approximations

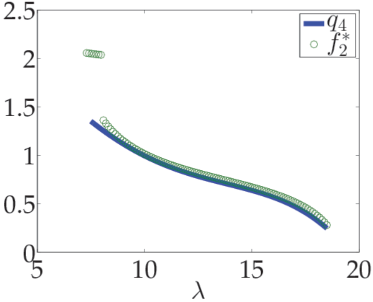
Moment-SOS approach [Lasserre 10]:

$$(D_k) \begin{cases} \max\limits_{q \in \mathbb{R}_{2k}[\lambda]} & \sum\limits_{i=0}^{2k} q_i / (1+i) \\ \text{s.t.} & f_2(\mathbf{x}) - q(\lambda) \in \mathcal{Q}_{2k}(\mathbf{K}) \end{cases} .$$

- The hierarchy (D_k) provides a sequence (q_k) of **polynomial under-approximations** of $f^*(\lambda)$.
- $\lim_{d \to \infty} \int_0^1 (f^*(\lambda) - q_k(\lambda)) d\lambda = 0$

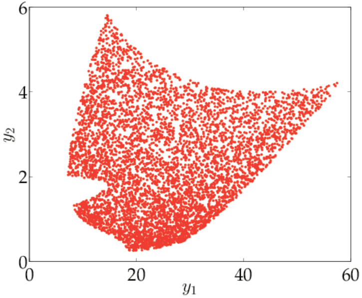
A Hierarchy of Polynomial Approximations

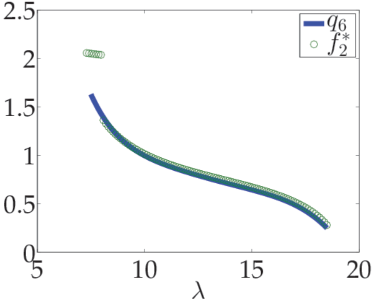




Degree 4

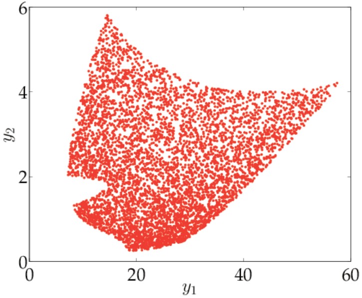
A Hierarchy of Polynomial Approximations

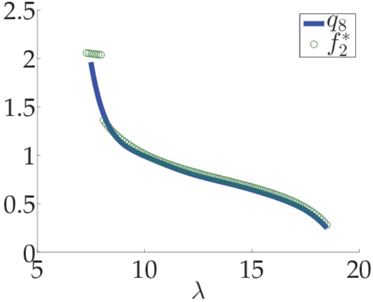




Degree 6

A Hierarchy of Polynomial Approximations





Degree 8

Contributions

- Numerical schemes that **avoid computing finitely many points**.

- Pareto curve approximation with polynomials, **convergence guarantees** in L_1-norm

V. Magron, D. Henrion, J.B. Lasserre. Approximating Pareto Curves using Semidefinite Relaxations. *Operations Research Letters*. arxiv:1404.4772, April 2014.

Polynomial Images of Semialgebraic Sets

- Semialgebraic set $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geqslant 0, \ldots, g_l(\mathbf{x}) \geqslant 0\}$

- A polynomial map $f : \mathbb{R}^n \to \mathbb{R}^m$,
  $\mathbf{x} \mapsto f(\mathbf{x}) := (f_1(\mathbf{x}), \ldots, f_m(\mathbf{x}))$

- $\deg f = d := \max\{\deg f_1, \ldots, \deg f_m\}$

- $\mathbf{F} := f(\mathbf{S}) \subseteq \mathbf{B}$, with $\mathbf{B} \subset \mathbb{R}^m$ a box or a ball

- Tractable approximations of $\mathbf{F}$?

Polynomial Images of Semialgebraic Sets

- Includes important special cases:

1. $m = 1$: **polynomial optimization**

$$\mathbf{F} \subseteq [\inf_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x}), \sup_{\mathbf{x} \in \mathbf{S}} f(\mathbf{x})]$$

2. Approximate **projections** of $\mathbf{S}$ when $f(\mathbf{x}) := (x_1, \ldots, x_m)$

3. **Pareto curve** approximations

   For f_1,f_2 two conflicting criteria: $(\mathbf{P}) \left\{ \min_{\mathbf{x} \in \mathbf{S}} (f_1(\mathbf{x})\, f_2(\mathbf{x}))^\top \right\}$

Method 1: Existential Quantifier Elimination

Another point of view:

$$\mathbf{F} = \{\mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } f(\mathbf{x}) = \mathbf{y}\} \ ,$$

Method 1: Existential Quantifier Elimination

Another point of view:

$$\mathbf{F} = \{\mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } \|\mathbf{y} - f(\mathbf{x})\|_2^2 = 0\} \ ,$$

Method 1: Existential Quantifier Elimination

Another point of view:

$$\mathbf{F} = \{\mathbf{y} \in \mathbf{B} : \exists \mathbf{x} \in \mathbf{S} \text{ s.t. } h_f(\mathbf{x}, \mathbf{y}) \geqslant 0\} \ ,$$

with

$$h_f(\mathbf{x}, \mathbf{y}) := -\|\mathbf{y} - f(\mathbf{x})\|_2^2 \ .$$

Method 1: Existential Quantifier Elimination

Existential QE: approximate $\mathbf{F}$ as closely as desired [Lasserre 14]

$$\mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geqslant 0\} \ ,$$

for some polynomials $q_k \in \mathbb{R}_{2k}[\mathbf{y}]$.

Method 1: Outer Approximations of F

- Let $\mathbf{K} = \mathbf{S} \times \mathbf{B}$, $\mathcal{Q}_k(\mathbf{K})$ be the k-truncated quadratic module

- **REMEMBER**:
  $q - h_f \in \mathcal{Q}_k(\mathbf{K}) \Longrightarrow \forall (\mathbf{x}, \mathbf{y}) \in \mathbf{K}, q(\mathbf{y}) - h_f(\mathbf{x}, \mathbf{y}) \geqslant 0$

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- Define $h(\mathbf{y}) := \sup_{\mathbf{x} \in \mathbf{S}} h_f(\mathbf{x}, \mathbf{y})$

- Hierarchy of Semidefinite programs:

$$\inf_q \Big\{ \int_{\mathbf{B}} (q - h) d\mathbf{y} : q - h_f \in \mathcal{Q}_k(\mathbf{K})) \Big\} \ .$$

Method 1: Outer Approximations of F

Assuming the existence of solution q_k, the sublevel sets

$$\mathbf{F}_k^1 := \{\mathbf{y} \in \mathbf{B} : q_k(\mathbf{y}) \geqslant 0\} \supseteq \mathbf{F} \ ,$$

provide a sequence of certified outer approximations of $\mathbf{F}$.

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It comes from the following:

- q_k feasible solution, $q_k - h_f \in \mathcal{Q}_k(\mathbf{K})$
- $\forall (\mathbf{x}, \mathbf{y}) \in \mathbf{K}, q_k(\mathbf{y}) \geqslant h_f(\mathbf{x}, \mathbf{y}) \Longleftrightarrow \forall \mathbf{y}, q_k(\mathbf{y}) \geqslant h(\mathbf{y}) \ .$

Method 1: Strong Convergence Property

Theorem

Assuming that $\mathring{\mathbf{S}} \neq \emptyset$ and $\mathcal{Q}_k(\mathbf{K})$ is Archimedean,

1. The sequence of optimal solutions (q_k) converges to h w.r.t the $L_1(\mathbf{B})$-norm:

$$\lim_{k \to \infty} \int_{\mathbf{B}} |q_k - h| d\mathbf{y} = 0 \quad , (q_k \to_{L_1} h)$$

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2. $$\lim_{k\to\infty} \mathrm{vol}(\mathbf{F}_k^1 \backslash \mathbf{F}) = 0 \ .$$

Method 2: Support of Image Measures

- **Pushforward** $f_{\#} : \mathcal{M}(\mathbf{S}) \to \mathcal{M}(\mathbf{B})$:

$$f_{\#}\mu_0(\mathbf{A}) := \mu_0(\{\mathbf{x} \in \mathbf{S} : f(\mathbf{x}) \in \mathbf{A}\}), \quad \forall \mathbf{A} \in \mathcal{B}(\mathbf{B}), \forall \mu_0 \in \mathcal{M}(\mathbf{S})$$

- $f_{\#}\mu_0$ is the **image measure** of μ_0 under f

Method 2: Support of Image Measures

$$
\begin{aligned}
p^* := \sup_{\mu_0, \mu_1, \hat{\mu}_1} \quad & \int_{\mathbf{B}} \mu_1 \\
\text{s.t.} \quad & \mu_1 + \hat{\mu}_1 = \lambda_{\mathbf{B}} \,, \\
& \mu_1 = f_{\#} \mu_0 \,, \\
& \mu_0 \in \mathcal{M}_+(\mathbf{S}), \quad \mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B}) \,.
\end{aligned}
$$

Lebesgue measure on $\mathbf{B}$ is $\lambda_{\mathbf{B}}(d\mathbf{y}) := \mathbf{1}_{\mathbf{B}}(\mathbf{y}) \, d\mathbf{y}$

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\end{aligned}
$$

Lemma

Let μ_1^* be an optimal solution of the above LP.
Then $\mu_1^* = \lambda_{\mathbf{F}}$ and $p^* = \operatorname{vol} \mathbf{F}$.

Method 2: Primal-dual LP Formulation

Primal LP

$$
\begin{aligned}
p^* := \sup_{\mu_0,\mu_1,\hat{\mu}_1} \quad & \int \mu_1 \\
\text{s.t.} \quad & \mu_1 + \hat{\mu}_1 = \lambda_{\mathbf{B}} \,, \\
& \mu_1 = f_{\#}\mu_0 \,, \\
& \mu_0 \in \mathcal{M}_+(\mathbf{S}) \,, \\
& \mu_1, \hat{\mu}_1 \in \mathcal{M}_+(\mathbf{B}) \,.
\end{aligned}
$$

Dual LP

$$
\begin{aligned}
d^* := \inf_{v,w} \quad & \int w(\mathbf{y}) \, \lambda_{\mathbf{B}}(d\mathbf{y}) \\
\text{s.t.} \quad & v(f(\mathbf{x})) \geqslant 0, \quad \forall \mathbf{x} \in \mathbf{S} \,, \\
& w(\mathbf{y}) \geqslant 1 + v(\mathbf{y}), \quad \forall \mathbf{y} \in \mathbf{B} \,, \\
& w(\mathbf{y}) \geqslant 0, \quad \forall \mathbf{y} \in \mathbf{B} \,, \\
& v, w \in \mathcal{C}(\mathbf{B}) \,.
\end{aligned}
$$

Method 2: Strong Convergence Property

Strengthening of the dual LP:

$$
\begin{aligned}
d_k^* := \inf_{v,w} \quad & \sum_{\beta \in \mathbb{N}_{2k}^m} w_\beta z_\beta^{\mathbf{B}} \\
\text{s.t.} \quad & v \circ f \in \mathcal{Q}_{kd}(\mathbf{S}), \\
& w - 1 - v \in \mathcal{Q}_k(\mathbf{B}), \\
& w \in \mathcal{Q}_k(\mathbf{B}), \\
& v, w \in \mathbb{R}_{2k}[\mathbf{y}].
\end{aligned}
$$

Method 2: Strong Convergence Property

Theorem

Assuming that $\mathring{\mathbf{F}} \neq \emptyset$ and $\mathcal{Q}_k(\mathbf{S})$ is Archimedean,

1. The sequence (w_k) converges to $\mathbf{1}_{\mathbf{F}}$ w.r.t the $L_1(\mathbf{B})$-norm:

$$\lim_{k \to \infty} \int_{\mathbf{B}} |w_k - \mathbf{1}_{\mathbf{F}}| d\mathbf{y} = 0 \ .$$

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$$\lim_{k \to \infty} \int_{\mathbf{B}} |w_k - \mathbf{1_F}| d\mathbf{y} = 0 \ .$$

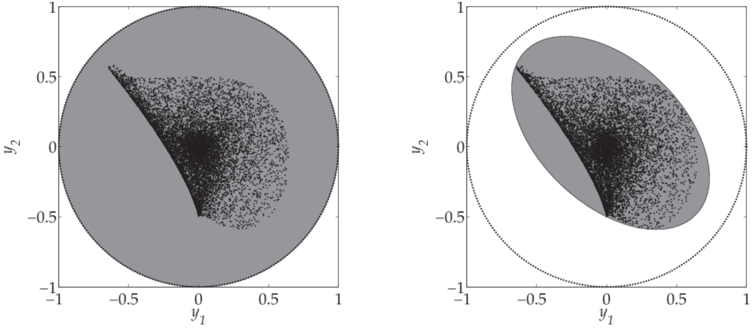
2. Let $\mathbf{F}_k^2 := \{\mathbf{y} \in \mathbf{B} : w_k(\mathbf{y}) \geqslant 1\}$. Then,

$$\lim_{k \to \infty} \mathrm{vol}(\mathbf{F}_k^2 \backslash \mathbf{F}) = 0 \ .$$

Polynomial Image of the Unit Ball

Image of the unit ball $\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\|_2^2 \leqslant 1\}$ by

$$f(\mathbf{x}) := (x_1 + x_1 x_2, x_2 - x_1^3)/2$$



$\mathbf{F}_1^1$ $\mathbf{F}_1^2$

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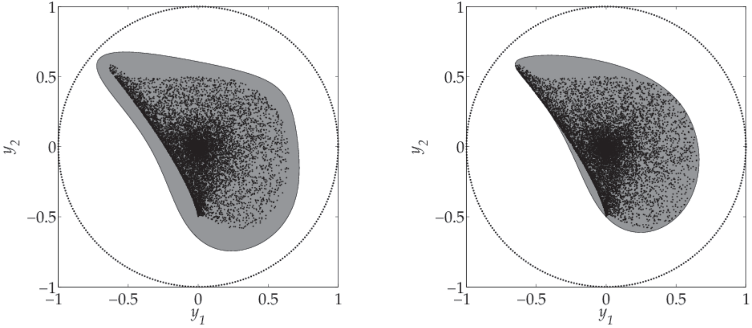

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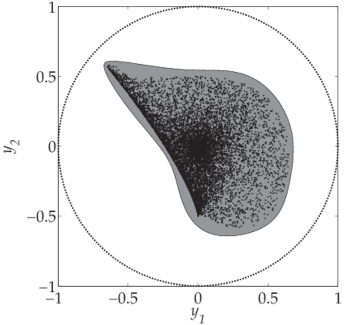
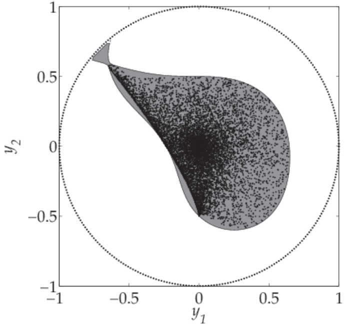


$\mathbf{F}_3^1$



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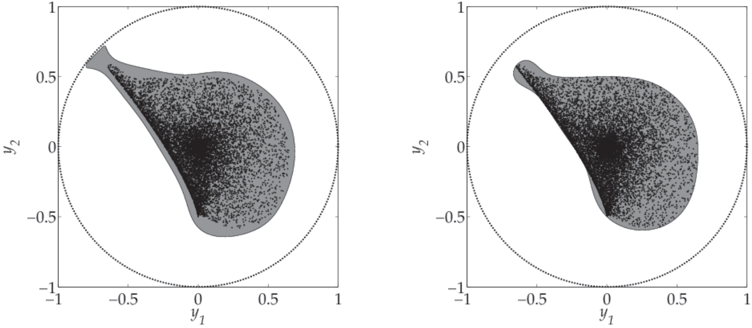
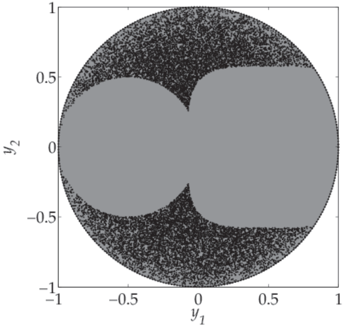


$\mathbf{F}_4^1$ $\mathbf{F}_4^2$

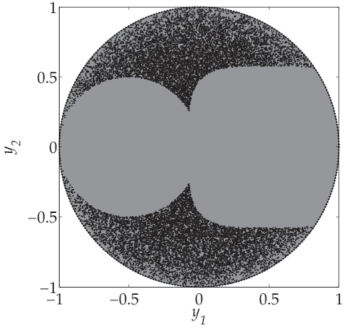
Semialgebraic Set Projections

$f(\mathbf{x}) = (x_1, x_2)$: projection on $\mathbb{R}^2$ of the semialgebraic set

$$\mathbf{S} := \{\mathbf{x} \in \mathbb{R}^3 : \|\mathbf{x}\|_2^2 \leqslant 1, 1/4 - (x_1 + 1/2)^2 - x_2^2 \geqslant 0, \\ 1/9 - (x_1 - 1/2)^4 - x_2^4 \geqslant 0\}$$



$\mathbf{F}_2^1$



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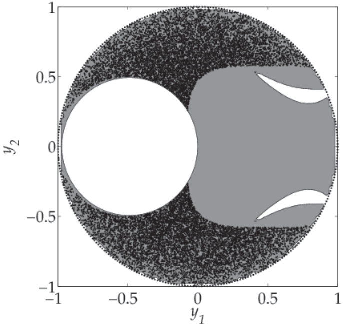
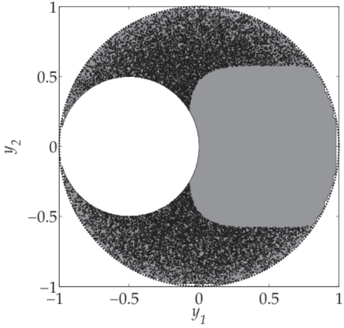


$\mathbf{F}_3^1$



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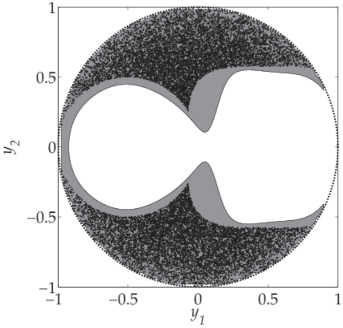
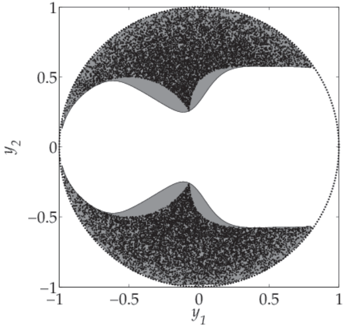


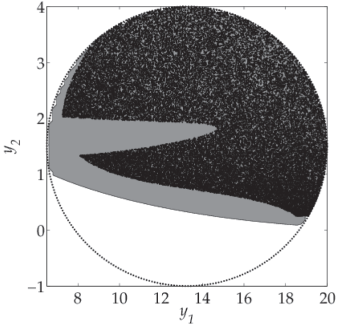
$\mathbf{F}_4^1$



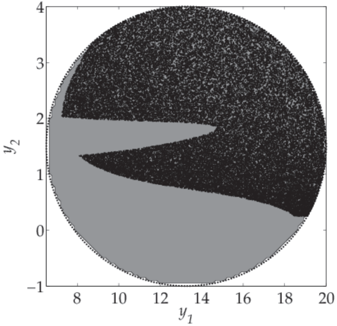
$\mathbf{F}_4^2$

Approximating Pareto Curves

Back on our previous nonconvex example:



$\mathbf{F}_1^1$



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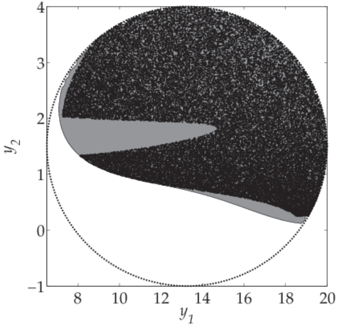
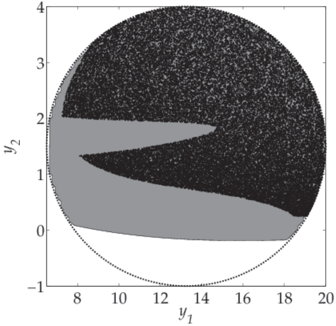


$\mathbf{F}_2^1$



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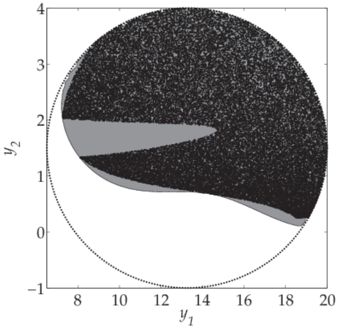
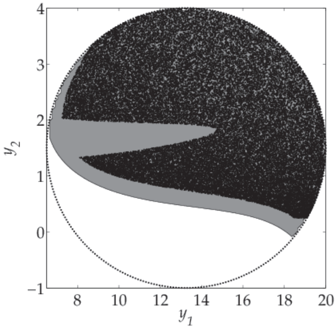


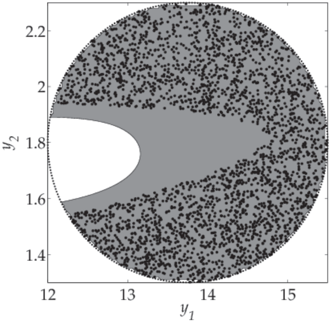
$\mathbf{F}_3^1$



$\mathbf{F}_3^2$

Approximating Pareto Curves

"Zoom" on the region which is hard to approximate:



$\mathbf{F}_4^1$

Approximating Pareto Curves

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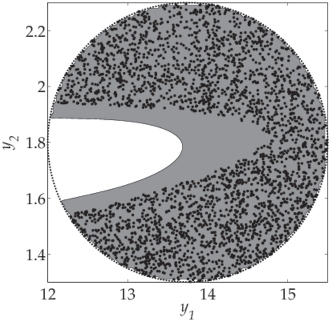

$\mathbf{F}_5^1$

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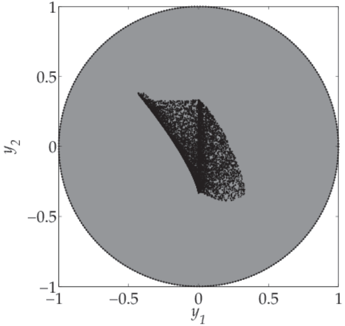
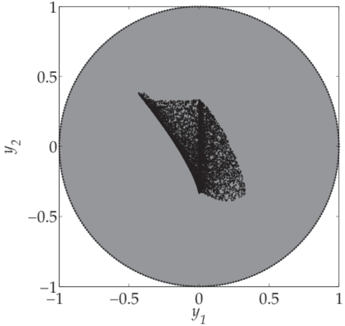


$\mathbf{F}_1^1$



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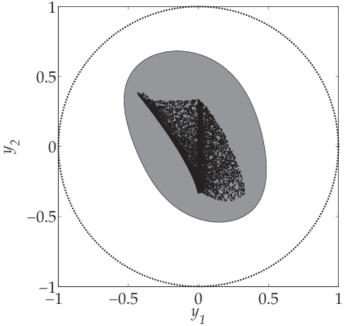
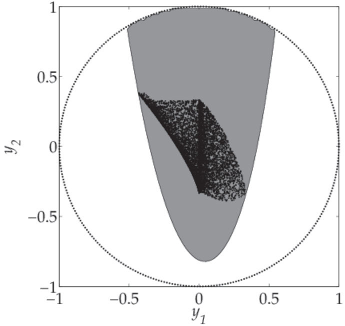


$\mathbf{F}_2^1$



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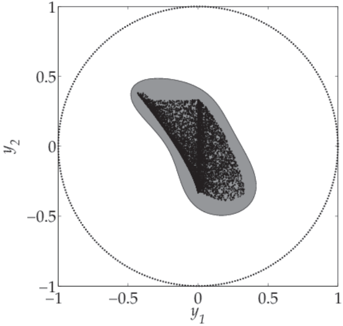
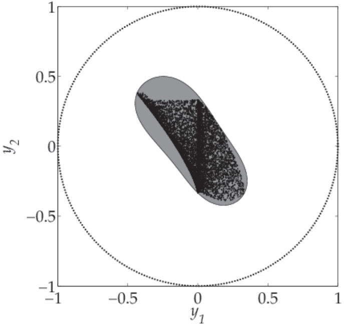


$\mathbf{F}_3^1$



$\mathbf{F}_3^2$

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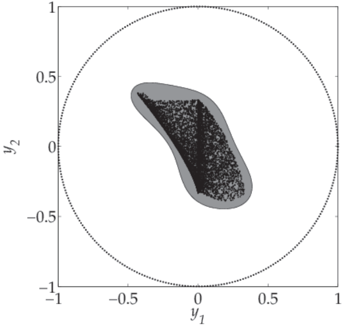
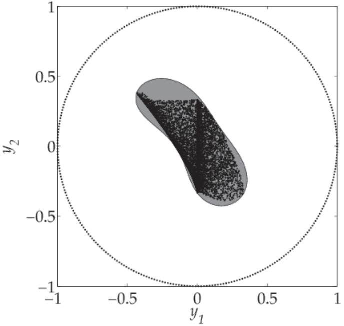


$\mathbf{F}_4^1$



$\mathbf{F}_4^2$

Contributions

V. Magron, D. Henrion, J.B. Lasserre. Semidefinite approximations of projections and polynomial images of semialgebraic sets. oo:2014.10.4606, October 2014.

Polynomial Programs (One-loop with Guards)

- $r, s, T^i, T^e \in \mathbb{R}[\mathbf{x}]$
- $\mathbf{x}_0 \in \mathbf{X}_0$, with $\mathbf{X}_0$ semialgebraic set

```
x = x0;
while (r(x) <= 0){
   if (s(x) <= 0){
         x = T^i(x);
         }
   else{
         x = T^e(x);
         }
}
```

Polynomial Inductive Invariants

Sufficient condition to get inductive invariant:

$$\alpha := \min_{q \in \mathbb{R}[\mathbf{x}]} \quad \sup_{\mathbf{x} \in \mathbf{X}_0} q(\mathbf{x})$$

$$\text{s.t.} \quad q - q \circ T^i \geqslant 0 \ , \text{ if } s(\mathbf{x}) \leqslant 0 \ ,$$

$$q - q \circ T^e \geqslant 0 \ , \text{ if } s(\mathbf{x}) \geqslant 0 \ ,$$

$$q - \kappa \geqslant 0 \ .$$

- $\bigcup_{k \in \mathbb{N}} \mathbf{X}_k \subseteq \{\mathbf{x} \in \mathbb{R}^n : q(\mathbf{x}) \leqslant \alpha\} \subseteq \{\mathbf{x} \in \mathbb{R}^n : \kappa(\mathbf{x}) \leqslant \alpha\}$

Bounding Polynomial Invariants

Sufficient condition to get bounding inductive invariant:

$$\alpha := \min_{q \in \mathbb{R}[\mathbf{x}]} \quad \sup_{\mathbf{x} \in \mathbf{X}_0} q(\mathbf{x})$$

$$\text{s.t.} \quad q - q \circ T^i \geqslant 0 \;, \text{ if } s(\mathbf{x}) \leqslant 0 \;,$$

$$q - q \circ T^e \geqslant 0 \;, \text{ if } s(\mathbf{x}) \geqslant 0 \;,$$

$$q - \| \cdot \|_2^2 \geqslant 0 \;.$$

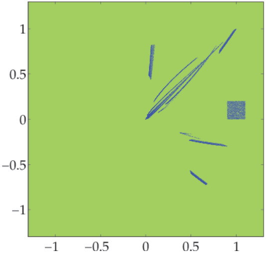
- $\bigcup_{k \in \mathbb{N}} \mathbf{X}_k \subseteq \{\mathbf{x} \in \mathbb{R}^n : q(\mathbf{x}) \leqslant \alpha\} \subseteq \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|^2 \leqslant \alpha\}$

Bounds for $\bigcup_{k \in \mathbb{N}} \mathbf{X}_k$

$\mathbf{X}_0 := [0.9, 1.1] \times [0, 0.2] \quad r(\mathbf{x}) := 1 \quad s(\mathbf{x}) := 1 - \|\mathbf{x}\|^2$

$T^i(\mathbf{x}) := (x_1^2 + x_2^3, x_1^3 + x_2^2) \quad T^e(\mathbf{x}) := (\frac{1}{2}x_1^2 + \frac{2}{5}x_2^3, -\frac{3}{5}x_1^3 + \frac{3}{10}x_2^2)$

$\kappa(\mathbf{x}) = \|\mathbf{x}\|^2$



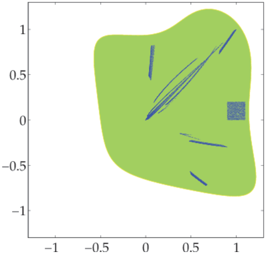
Degree 6

Bounds for $\bigcup_{k\in\mathbb{N}} \mathbf{X}_k$

$\mathbf{X}_0 := [0.9, 1.1] \times [0, 0.2] \quad r(\mathbf{x}) := 1 \quad s(\mathbf{x}) := 1 - \|\mathbf{x}\|^2$

$T^i(\mathbf{x}) := (x_1^2 + x_2^3, x_1^3 + x_2^2) \quad T^e(\mathbf{x}) := (\frac{1}{2}x_1^2 + \frac{2}{5}x_2^3, -\frac{3}{5}x_1^3 + \frac{3}{10}x_2^2)$

$\kappa(\mathbf{x}) = \|\mathbf{x}\|^2$



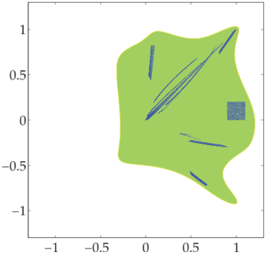
Degree 8

Bounds for $\bigcup_{k \in \mathbb{N}} \mathbf{X}_k$

$\mathbf{X}_0 := [0.9, 1.1] \times [0, 0.2] \quad r(\mathbf{x}) := 1 \quad s(\mathbf{x}) := 1 - \|\mathbf{x}\|^2$

$T^i(\mathbf{x}) := (x_1^2 + x_2^3, x_1^3 + x_2^2) \quad T^e(\mathbf{x}) := (\frac{1}{2}x_1^2 + \frac{2}{5}x_2^3, -\frac{3}{5}x_1^3 + \frac{3}{10}x_2^2)$

$\kappa(\mathbf{x}) = \|\mathbf{x}\|^2$



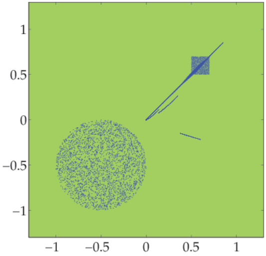
Degree 10

Does $\bigcup_{k\in\mathbb{N}} \mathbf{X}_k$ avoid unsafe region?

$$\mathbf{X}_0 := [0.5, 0.7]^2 \quad r(\mathbf{x}) := 1 \quad s(\mathbf{x}) := 1 - \|\mathbf{x}\|^2$$

$$T^i(\mathbf{x}) := (x_1^2 + x_2^3, x_1^3 + x_2^2) \quad T^e(\mathbf{x}) := (\frac{1}{2}x_1^2 + \frac{2}{5}x_2^3, -\frac{3}{5}x_1^3 + \frac{3}{10}x_2^2)$$

$$\kappa(\mathbf{x}) = \frac{1}{4} - (x_1 + \frac{1}{2})^2 - (x_2 + \frac{1}{2})^2$$



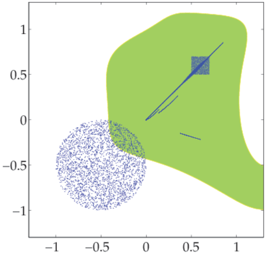
Degree 6

Does $\bigcup_{k\in\mathbb{N}} \mathbf{X}_k$ avoid unsafe region?

$\mathbf{X}_0 := [0.5, 0.7]^2 \quad r(\mathbf{x}) := 1 \quad s(\mathbf{x}) := 1 - \|\mathbf{x}\|^2$

$T^i(\mathbf{x}) := (x_1^2 + x_2^3, x_1^3 + x_2^2) \quad T^e(\mathbf{x}) := (\frac{1}{2}x_1^2 + \frac{2}{5}x_2^3, -\frac{3}{5}x_1^3 + \frac{3}{10}x_2^2)$

$\kappa(\mathbf{x}) = \frac{1}{4} - (x_1 + \frac{1}{2})^2 - (x_2 + \frac{1}{2})^2$



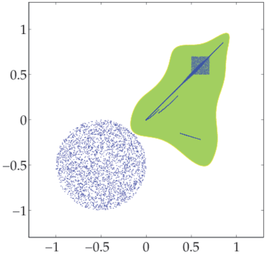
Degree 8

Does $\bigcup_{k\in\mathbb{N}} \mathbf{X}_k$ avoid unsafe region?

$\mathbf{X}_0 := [0.5, 0.7]^2 \quad r(\mathbf{x}) := 1 \quad s(\mathbf{x}) := 1 - \|\mathbf{x}\|^2$

$T^i(\mathbf{x}) := (x_1^2 + x_2^3, x_1^3 + x_2^2) \quad T^e(\mathbf{x}) := (\frac{1}{2}x_1^2 + \frac{2}{5}x_2^3, -\frac{3}{5}x_1^3 + \frac{3}{10}x_2^2)$

$\kappa(\mathbf{x}) = \frac{1}{4} - (x_1 + \frac{1}{2})^2 - (x_2 + \frac{1}{2})^2$



Degree 10

Contributions

A. Adjé, V. Magron. Polynomial template generation using sum-of-squares programming. *Technical Report*. arxiv:1409.3941, October 2014.

Ongoing: Bounding Floating-point Errors

- Exact:

$$f(\mathbf{x}) := x_1 x_2 + x_3 x_4$$

- Floating-point:

$$\hat{f}(\mathbf{x}, \boldsymbol{\epsilon}) := [x_1 x_2 (1 + \epsilon_1) + x_3 x_4 (1 + \epsilon_2)](1 + \epsilon_3)$$

- $\mathbf{x} \in \mathbf{S}$, $\quad \mid \epsilon_i \mid \leqslant 2^{-p} \quad p = 24$ (single) or 53 (double)

Ongoing: Bounding Floating-point Errors

Input: exact $f(\mathbf{x})$, floating-point $\hat{f}(\mathbf{x}, \boldsymbol{\epsilon})$, $\mathbf{x} \in \mathbf{S}$, $| \epsilon_i | \leqslant 2^{-p}$
Output: Bounds for $f - \hat{f}$

1: Error $r(\mathbf{x}, \boldsymbol{\epsilon}) := f(\mathbf{x}) - \hat{f}(\mathbf{x}, \boldsymbol{\epsilon}) = \sum_{\boldsymbol{\alpha}} r_{\boldsymbol{\alpha}}(\boldsymbol{\epsilon}) \mathbf{x}^{\boldsymbol{\alpha}}$

2: Decompose $r(\mathbf{x}, \boldsymbol{\epsilon}) = l(\mathbf{x}, \boldsymbol{\epsilon}) + h(\mathbf{x}, \boldsymbol{\epsilon})$, l **linear** in $\boldsymbol{\epsilon}$

3: Bound $h(\mathbf{x}, \boldsymbol{\epsilon})$ with interval arithmetic

4: Bound $l(\mathbf{x}, \boldsymbol{\epsilon})$ with **SPARSE SUMS OF SQUARES**

Conclusion

With **MOMENT-SOS HIERARCHIES**, you can

- Optimize nonlinear (transcendental) functions

- Approximate Pareto Curves, images and projections of semialgebraic sets

- Analyze programs

Conclusion

Further research:

- Alternative polynomial bounds using geometric programming (T. de Wolff, S. Iliman)
- Mixed LP/SOS certificates (trade-off CPU/precision)

End

Thank you for your attention!

`cas.ee.ic.ac.uk/people/vmagron`

Hidden Details

$$\ell_{\mathbf{z}}(q) : q \in \mathbb{R}[\mathbf{x}] \mapsto \sum_{\alpha} q_{\alpha} \mathbf{z}_{\alpha}$$

- Moment matrix
  $\mathbf{M}(\mathbf{z})_{\mathbf{x}^{\alpha},\mathbf{x}^{\beta}} := \ell_{\mathbf{z}}(\mathbf{x}^{\alpha}\,\mathbf{x}^{\beta}) = \mathbf{z}_{\alpha+\beta}$

- Localizing matrix $\mathbf{M}(g_j\,\mathbf{z})$ associated with g_j
  $\mathbf{M}(g_j\,\mathbf{z})_{\mathbf{x}^{\alpha},\mathbf{x}^{\beta}} := \ell_{\mathbf{z}}(g_j\,\mathbf{x}^{\alpha}\,\mathbf{x}^{\beta}) = \sum_{\gamma} g_{j,\gamma}\,\mathbf{z}_{\alpha+\beta+\gamma}$

Hidden Details

- $\mathbf{M}_k(\mathbf{z})$ contains $\binom{n+2k}{n}$ variables, has size $\binom{n+k}{n}$
- Truncated matrix of order $k = 2$ with variables x_1, x_2:

$$
\begin{array}{c}
\\ \\ \\ \\ \mathbf{M}_2(\mathbf{z}) =
\end{array}
\begin{array}{c|c}
 & \begin{array}{c|cc|ccc} 1 & x_1 & x_2 & x_1^2 & x_1x_2 & x_2^2 \end{array} \\
\begin{array}{c} 1 \\ \hline x_1 \\ x_2 \\ \hline x_1^2 \\ x_1x_2 \\ x_2^2 \end{array} &
\left(\begin{array}{c|cc|ccc}
1 & z_{1,0} & z_{0,1} & z_{2,0} & z_{1,1} & z_{0,2} \\ \hline
z_{1,0} & z_{2,0} & z_{1,1} & z_{3,0} & z_{2,1} & z_{1,2} \\
z_{0,1} & z_{1,1} & z_{0,2} & z_{2,1} & z_{1,2} & z_{0,3} \\ \hline
z_{2,0} & z_{3,0} & z_{2,1} & z_{4,0} & z_{3,1} & z_{2,2} \\
z_{1,1} & z_{2,1} & z_{1,2} & z_{3,1} & z_{2,2} & z_{1,3} \\
z_{0,2} & z_{1,2} & z_{0,3} & z_{2,2} & z_{1,3} & z_{0,4}
\end{array}\right)
\end{array}
$$

Hidden Details

- Consider $g_1(\mathbf{x}) := 2 - x_1^2 - x_2^2$. Then $v_1 = \lceil \deg g_1/2 \rceil = 1$.

$$\mathbf{M}_1(g_1\,\mathbf{z}) = \begin{array}{c} \\ 1 \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & x_1 & x_2 \end{array} \\ \begin{pmatrix} 2 - z_{2,0} - z_{0,2} & 2z_{1,0} - z_{3,0} - z_{1,2} & 2z_{0,1} - z_{2,1} - z_{0,3} \\ 2z_{1,0} - z_{3,0} - z_{1,2} & 2z_{2,0} - z_{4,0} - z_{2,2} & 2z_{1,1} - z_{3,1} - z_{1,3} \\ 2z_{0,1} - z_{2,1} - z_{0,3} & 2z_{1,1} - z_{3,1} - z_{1,3} & 2z_{0,2} - z_{2,2} - z_{0,4} \end{pmatrix} \end{array}$$

$$\begin{aligned} \mathbf{M}_1(g_1\,\mathbf{z})(3,3) = \ell(g_1(\mathbf{x}) \cdot x_2 \cdot x_2) &= \ell(2x_2^2 - x_1^2 x_2^2 - x_2^4) \\ &= 2z_{0,2} - z_{2,2} - z_{0,4} \end{aligned}$$

Hidden Details

- Truncation with moments of order at most $2k$
- $v_j := \lceil \deg g_j / 2 \rceil$
- Hierarchy of semidefinite relaxations:

$$\begin{cases} \inf_{\mathbf{z}} \ell_{\mathbf{z}}(f) & = & \sum_\alpha \int_{\mathbf{S}} f_\alpha \, \mathbf{x}^\alpha \, \mu(d\mathbf{x}) = \sum_\alpha f_\alpha \, \mathbf{z}_\alpha \\ \mathbf{M}_k(\mathbf{z}) & \succcurlyeq & 0\,, \\ \mathbf{M}_{k-v_j}(g_j \, \mathbf{z}) & \succcurlyeq & 0\,, \quad 1 \leqslant j \leqslant l, \\ \mathbf{z}_1 & = & 1\,. \end{cases}$$