Towards a Modern Floating-Point Environment

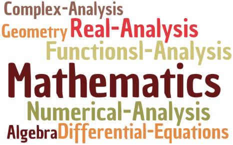
PhD thesis defense

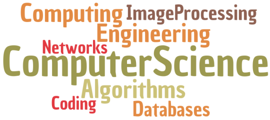
Olga Kupriianova

PEQUAN team, CalSci Dept, LIP6, UPMC

advisors: Jean-Claude Bajard, Christoph Lauter

What is Floating-Point Arithmetic





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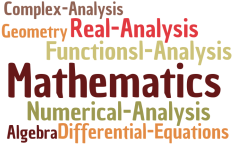

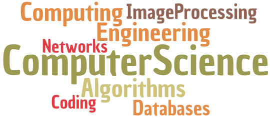

Continuous	**Discrete**
$\mathbb{R}$ numbers	Floating point (FP) numbers
$3/2 = 1.5$	$3/2 = 1.5e0$
$\pi = 3.14159{\color{red}2653589\dots}$	$\pi = 3.14159e0$
$\sqrt{2} = 1.41421{\color{red}3562373\dots}$	$\sqrt{2} = 1.41421e0$

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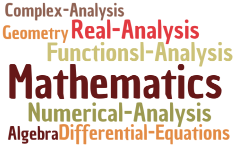

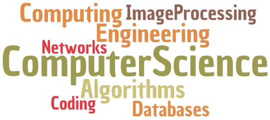

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$$\pm \underbrace{m_0.m_1m_2\ldots m_{k-1}}_{\text{significand}} \cdot \beta^E$$

E is exponent, k precision, β radix

Rounding

How to represent a real number in machine?

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$\pi \approx 3.14$	$e \approx 2.71$
$\pi \approx 3.141$	$e \approx 2.718$
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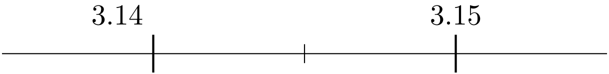

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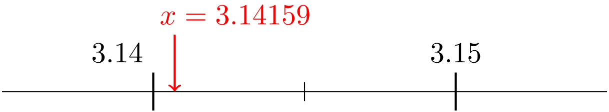

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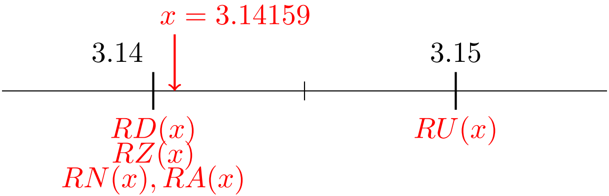

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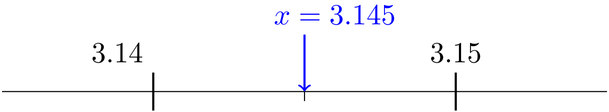

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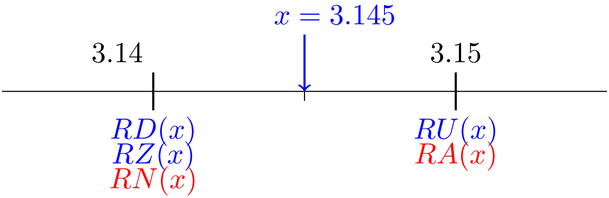
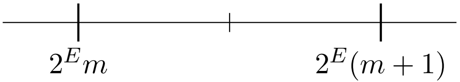


Accuracy and Rounding

Accuracy

Mathematical $x \to \hat{x}$ in FP arithmetic

Due to roundings, $|x - \hat{x}| > 0$



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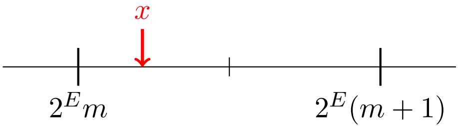

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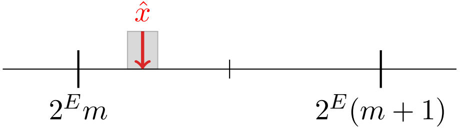

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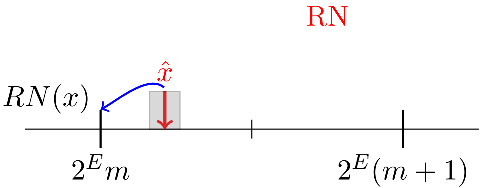

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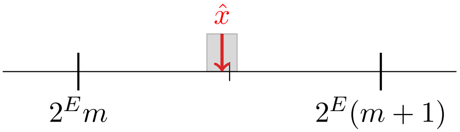

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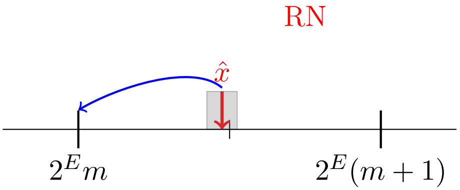

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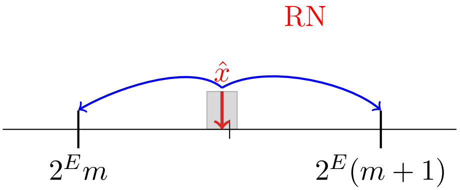

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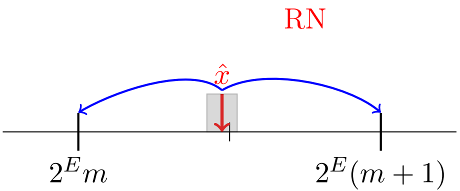

Table Maker's Dilemma (TMD)

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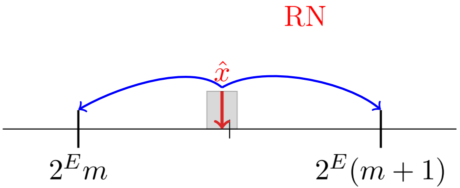

Table Maker's Dilemma (TMD)

How to determine the accuracy

of the approximation $\hat{x}$

Floating-Point Environment

- **IEEE754-1985**: binary FP numbers, rounding modes, arithmetic operations
  In total ~ 70 operations

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  mix binary and decimals in one operation
  provide more implementations of mathematical functions

Implementation of Mathematical Functions

Why do we need more implementations of mathematical functions?

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Mathematics

$$\exp(x) = e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n$$

$$\log_b(x) = y; \quad b^y = x$$

$$\operatorname{erf}(x) = \int_0^x e^{-t^2} dt$$

Computer Science

Several implementations for each:
exp - correctly rounded
exp - faithfully rounded
exp - with accuracy $\leq 2^{-45}$
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The existing libraries give *only few* implementations per function

Contributions/Structure

$$2^E m \quad 10^F n$$

$$2^E m \times 10^{F_1} n_1 = 10^{F_2} n_2$$

$$2^{E_1} m_1 + 10^F n = 2^{E_2} m_2$$

1. Mixed-Radix Arithmetic and Arbitrary Precision Base Conversion

2. Automatic Generation of Mathematical Functions (Metalibm)

$$\exp \quad \sin$$

$$\int_0^x \frac{\sin t}{t} dt$$

$$\log$$

Mixed-Radix Arithmetic

- Radix conversion algorithm for Mixed-Radix Arithmetic
- Correctly-rounded conversion from decimal character sequence to a binary FP number (scanf analogue)
- Research for fused multiply-add operation $xy \pm z$

Mixed-Radix Arithmetic

Radix Conversion

- Two-steps algorithm:
  1. Exponent computation is errorless
  2. Compute mantissa with specified accuracy
- Integer-based computations
- Memory consumption is known in advance

Mixed-Radix Arithmetic

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Mixed-Radix Arithmetic

Our scanf version

- User input is arbitrarily long
- No way to precompute the worst cases for the TMD
- Algorithm with integer computations, no memory allocations
- Two-ways scheme is used: fast easy rounding or slow hard
- Two conversions: decimal-binary and binary-decimal
- Gives correctly-rounded (CR) result for all the rounding modes

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Mixed-Radix Arithmetic

Mixed-radix FMA

$$xy \pm z$$

$$2^{-4} \cdot 139 \times 10^{-16} \cdot 346 - 2^3 \cdot 42 = 10^E m$$

- Addition and multiplication within *only* one rounding
- Worst cases search: reduce the iterations number with the use of continued fraction theory and variable relations
- Iterations reduced at $\sim 99\%$
- Still about $\sim 16\,000\,000\,000$ iterations
- First results obtained in ~ 10 weeks of computations

Code Generation for Mathematical Functions

1. Introduction & Motivation for Code Generation
2. Manual Function Implementation. Background
3. Metalibm: General Overview
4. Metalibm: Domain Splitting
5. Metalibm: Reconstruction for Vectorizable Code
6. Conclusion and Perspectives

Existing Mathematical Libraries (libm)

Existing implementations

- Intel's Math Kernel Libraries
- AMD's libm
- ARM's mathlib
- libmcr by Sun
- ...
- glibc libm
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How to get
$\exp(x)$ with 40 accurate bits
$\int_0^x \frac{\sin t}{t} dt$ with 25 bits

One Size Does not Fit All

Current request

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  ("quick and dirty", faithful, correctly-rounded)
- Several portability options (SIMD instructions)

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Who is going to write all these variants?

Task

Rough Computations

- 3 precisions from IEEE754
- ~ 50 functions in libm
- ~ 5 accuracies for each precision
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Generate parametrized implementations of mathematical functions
Metalibm

How to Implement a Mathematical Function

Elementary functions have transcendental values

$$\exp(1) = 2.71828182845904523536028747135266\ldots$$
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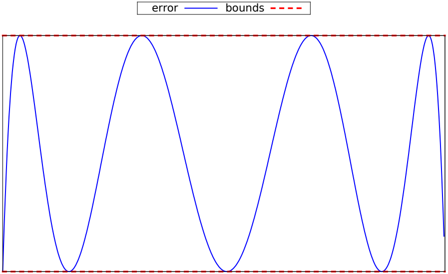

Approximation Polynomials

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Compute different approximations on $[0, 5]$

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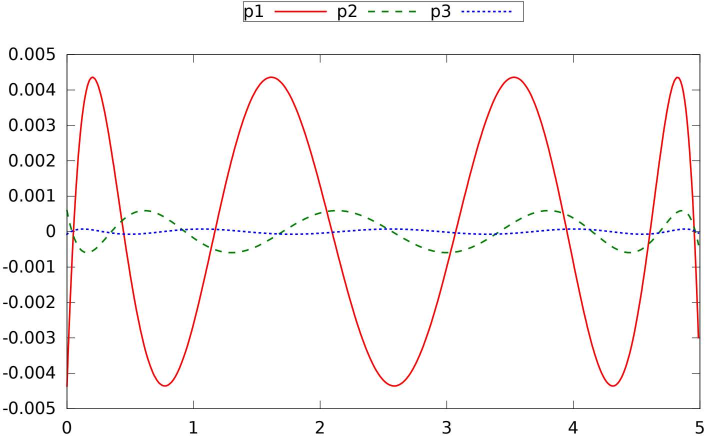

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Larger domain, larger degree

Conclusion: approximate *only* on small domains

Argument Reduction. Example

Example

Implement $f(x) = e^x$

Exploit the algebraic property $e^{a+b} = e^a \cdot e^b$

$$e^x = 2^{\frac{x}{\log 2}} = 2^{\left\lfloor \frac{x}{\log 2} \right\rceil} \cdot 2^{\frac{x}{\log 2} - \left\lfloor \frac{x}{\log 2} \right\rceil} = 2^E \cdot e^{x - E \log 2} = 2^E \cdot e^r$$

$$r \in \left[-\frac{\log(2)}{2}, \frac{\log(2)}{2} \right]$$

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P. Tang's table-based method

$$e^x = 2^m \cdot 2^{i/2^t} \cdot e^{r^*}, \quad r^* \in \left[-\frac{\log(2)}{2 \cdot 2^t}, \frac{\log(2)}{2 \cdot 2^t}\right]$$

$$m \in \mathbb{Z}, \quad t \in \mathbb{Z}, \quad 0 \le i \le 2^t - 1, \quad 2^{i/2^t} \text{ in table}$$

Argument Reduction. Does It Always Work?

Useful algebraic properties:

$$\exp(a+b) = \exp(a) \cdot \exp(b)$$
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$$u\rho'(u) + \rho(u-1) = 0, \quad \rho(u) = 1 \text{ for } 0 \le u \le 1$$

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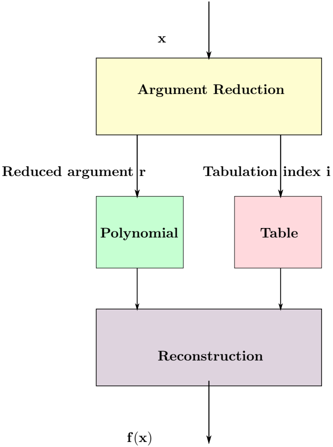
Piecewise-polynomial approximation

General Scheme



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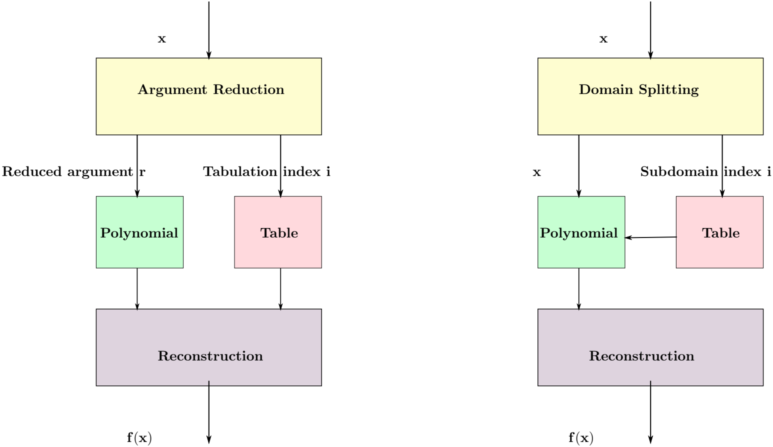
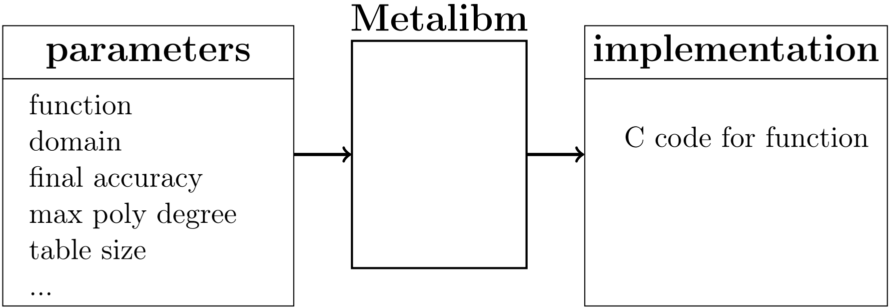


And filtering of special cases

Structure


Metalibm
parameters
function
domain
final accuracy
max poly degree
table size
...
implementation
C code for function

What is Metalibm

Academic Prototype

- Open-Source code generator for parametrized libm functions
- Part of ANR Metalibm Project
- Available at `http://metalibm.org/`

Objectives

- Push-button tool to implement functions $f : \mathbb{R} \to \mathbb{R}$
  - automatic argument reduction
  - automatic polynomial approximation
  - automatic domain splitting
- Support black-box functions
  - no function dictionary
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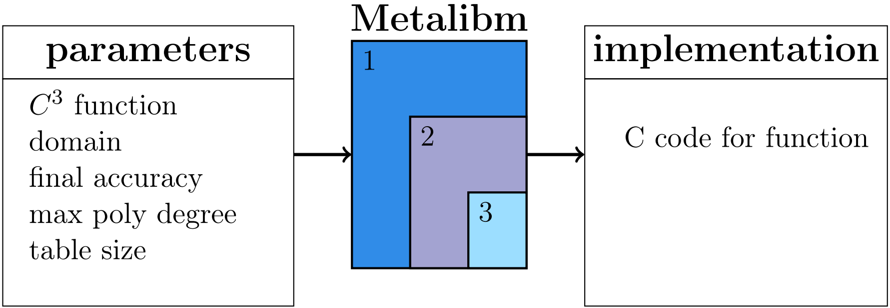
    $\Rightarrow$ use only C^3 functions that can be evaluated over intervals

Generation Levels



1 - Properties detection
2 - Domain splitting
3 - Approximation

Property Detection: Example on Exponential

Task

Generate $f(x)$ on $[a, b]$ with accuracy $\bar{\varepsilon}$

Generation hypothesis

$f(x)$ is of type β^x, unknown β

Property Detection: Example on Exponential

Task

Generate $f(x)$ on $[a, b]$ with accuracy $\bar{\varepsilon}$

Generation hypothesis

$f(x)$ is of type β^x, unknown β

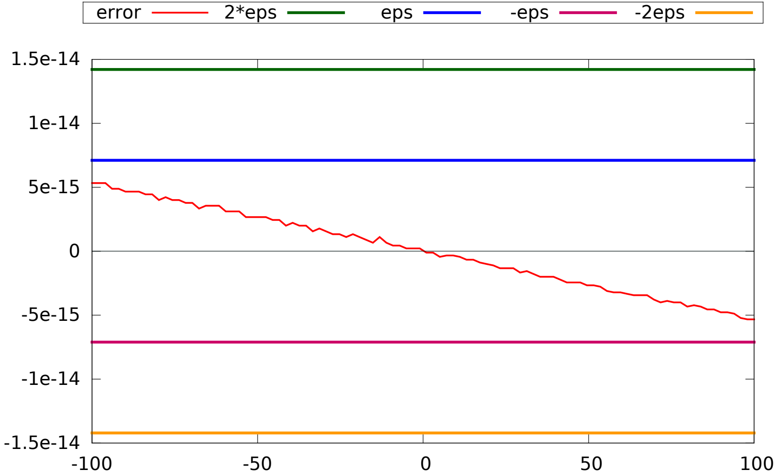
Finding the base

$\beta = \exp\left(\frac{\log(f(\xi))}{\xi}\right)$, for some $\xi \in [a, b]$

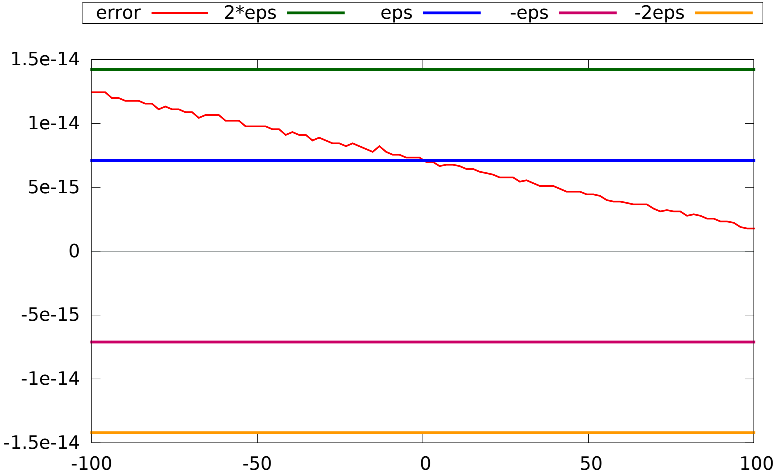
Decision of acceptance

$$\varepsilon = \left\| \frac{\beta^x}{f(x)} - 1 \right\|_\infty^{[a,b]} \qquad |\varepsilon| < |\bar{\varepsilon}|$$

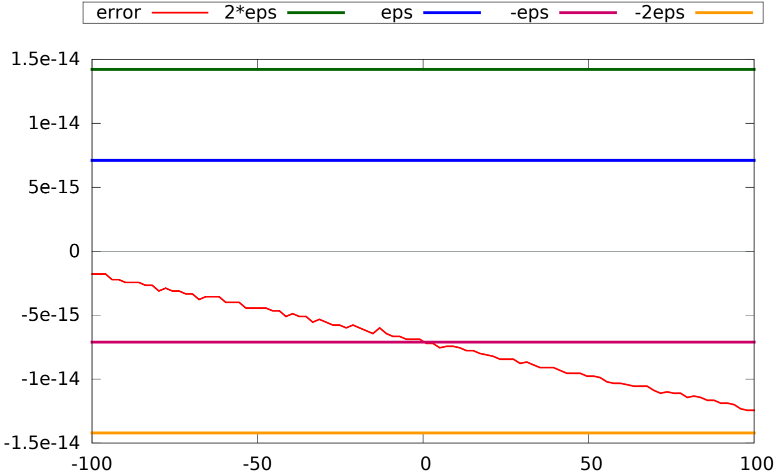
Property Detection: Example on Exponential



Property Detection: Example on Exponential



Property Detection: Example on Exponential



Property Detection

Currently detectable properties

- Exponential $f(a+b) = f(a)f(b)$
- Logarithmic $f(ab) = f(a) + f(b)$
- Periodic $f(x+C) = f(x)$
- Symmetric $f(x) = f(-x)$ and $f(x) = -f(x)$
- Sinh-like family $f(x) = \beta^x - \beta^{-x}$

When property detection does not work

Argument reduction

Step is based on mathematical properties:
$n^{a+b} = n^a \cdot n^b$, $\sin(x + 2\pi) = \sin(x)$, $\log(a \cdot b) = \log(a) + \log(b), \ldots$

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What properties do we know for $\mathrm{erf}(x)$, or a black-box function?

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When argument reduction does not work

Piecewise-polynomial approximation

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What properties do we know for $\mathrm{erf}(x)$, or a black-box function?

When argument reduction does not work

Piecewise-polynomial approximation

- Algorithm to split the domain
- Reconstruction becomes an execution of if-statements

Problem Statement

Task:
Split the domain $[a, b]$ into $I_0, I_1, \ldots, I_N$
Approximation degrees on I_k: $d_k \leq d_{\max}$

function	f
initial domain	$[a, b]$
accuracy	$\bar{\varepsilon}$
degree bound	$d_{\max}$

Problem Statement

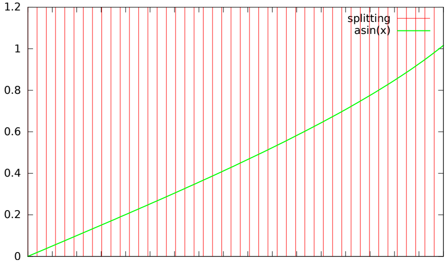
function	f
initial domain	$[a, b]$
accuracy	$\bar{\varepsilon}$
degree bound	$d_{\max}$

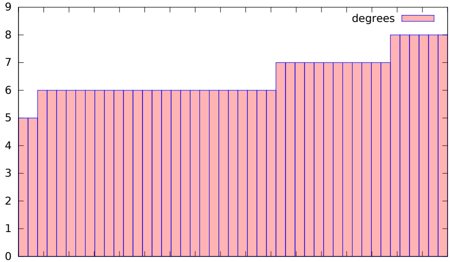
Task:
Split the domain $[a, b]$ into $I_0, I_1, \ldots, I_N$
Approximation degrees on I_k: $d_k \leq d_{\max}$

Naive Solution:
Split domain into N equal parts, N is large.

Problem Statement

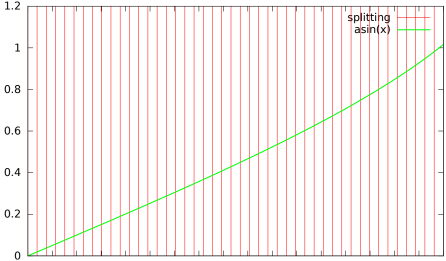
$$f = \mathrm{asin},\ [a,b] = [0, 0.85],\ d_{\max} \leq 8,\ \bar{\varepsilon} = 2^{-52}$$

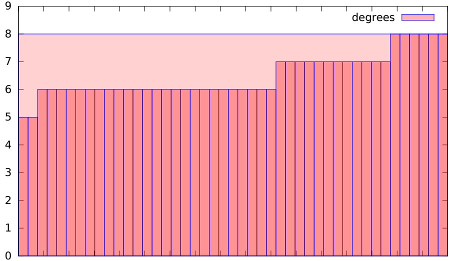




45 intervals

$$f = \text{asin},\ [a,b] = [0, 0.85],\ d_{\max} \leq 8,\ \bar{\varepsilon} = 2^{-52}$$





45 intervals

Problem Statement

The quantity of intervals $N \to \min$

Classic problem:

having f, $[a, b]$ and d
find a polynomial p and accuracy ε

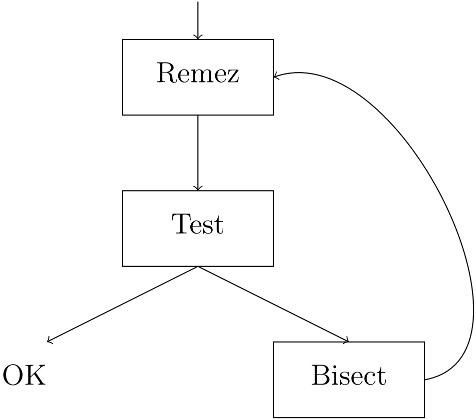
We need:

having f, $[a, b]$, bounds $\bar{\varepsilon}$ and $d_{\max}$
find I and polynomial p

Solution:

compute Remez polynomials and check the constraints?

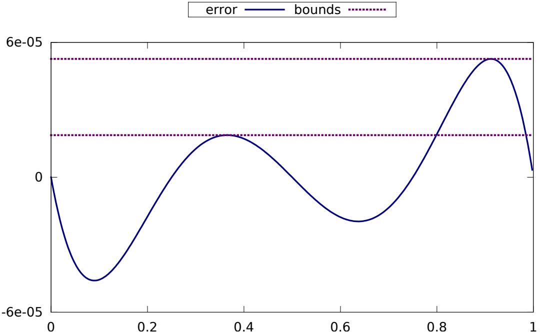
Problem Statement



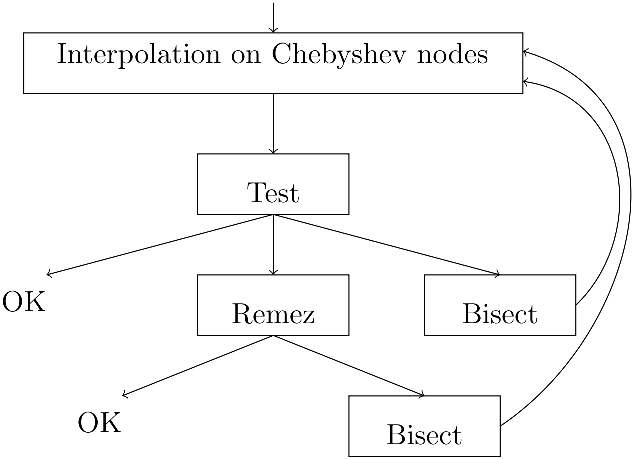
Domain Splitting

Theorem of de la Vallée-Poussin:

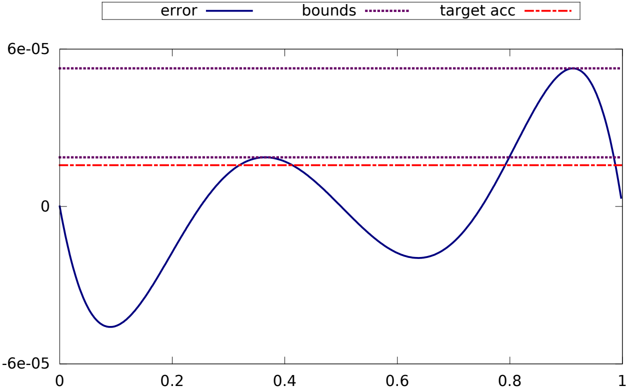
a continuous function f on $[a, b]$,
its approximating polynomial p
find the bounds for optimal error



Domain Splitting

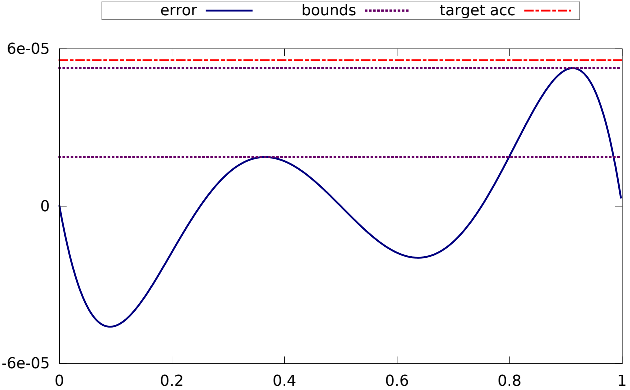


Domain Splitting



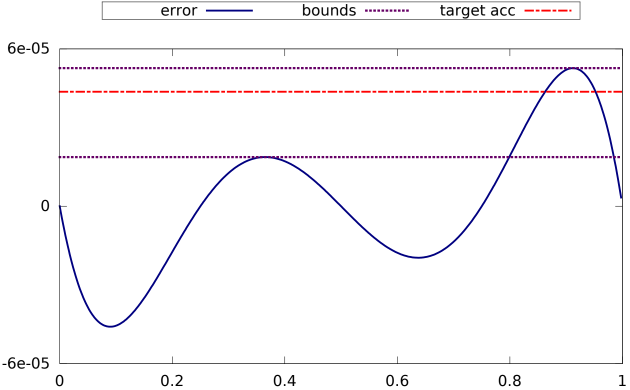
Need a split

Domain Splitting



No split

Domain Splitting

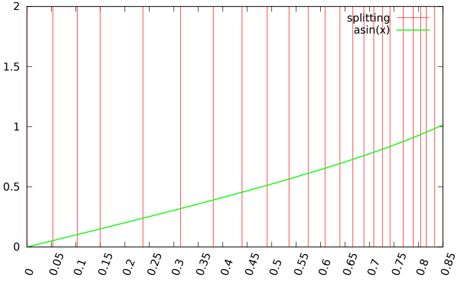


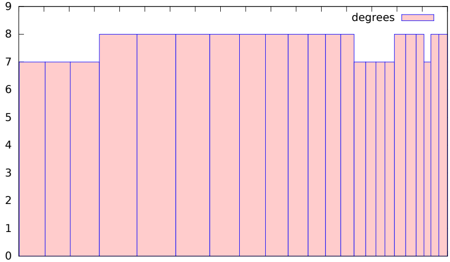
Need Remez

Domain Splitting. Our method

Bisection

$$f = \text{asin},\ [a,b] = [0, 0.85],\ d_{\max} \le 8,\ \bar{\varepsilon} = 2^{-52}$$

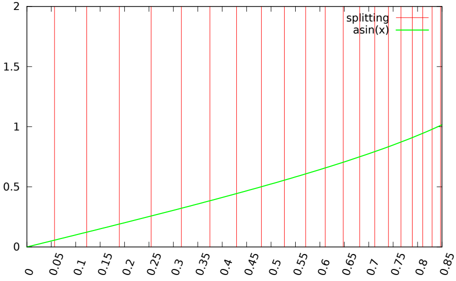


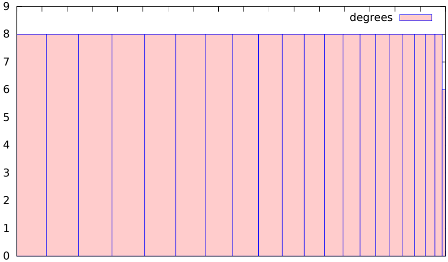


24 intervals

Our optimized method

$$f = \text{asin},\ [a,b] = [0, 0.85],\ d_{\max} \leq 8,\ \bar{\varepsilon} = 2^{-52}$$





18 intervals

Some results

measure	f_1	f_2	f_3	f_4
subdomains in bisection	24	15	9	12
subdomains in our method	18	10	5	8
subdomains saved	25%	30%	44 %	30%
coefficients saved	42	34	30	28
memory saved (bytes)	336	272	240	224
memory saved (%)	23%	35%	38.5%	45%

name	f	$\bar{\varepsilon}$	domain I	$d_{\max}$
f_1	asin	2^{-52}	$[0, 0.85]$	8
f_2	asin	2^{-45}	$[-0.75, 0.75]$	8
f_3	erf	2^{-51}	$[-0.75, 0.75]$	9
f_4	erf	2^{-45}	$[-0.75, 0.75]$	7

Problem Statement

Reconstruction gets tricky with arbitrary domain splitting:

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$f(x) \approx p_k(x)$, $x \in I_k$, $k \in [0, N] \cap \mathbb{Z}$

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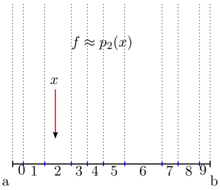


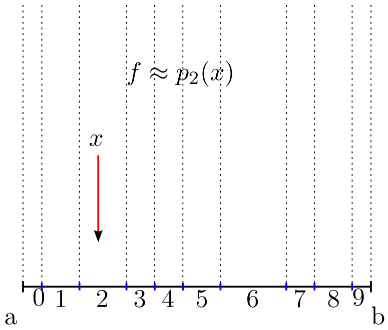
Problem Statement

```
/* compute i so that a[i] < x < a[i+1] */
        i=31;
        if (x < arctan_table[i][A].d) i-= 16;
        else i+=16;
        if (x < arctan_table[i][A].d) i-= 8;
        else i+= 8;
        if (x < arctan_table[i][A].d) i-= 4;
        else i+= 4;
        if (x < arctan_table[i][A].d) i-= 2;
        else i+= 2;
        if (x < arctan_table[i][A].d) i-= 1;
        else i+= 1;
        if (x < arctan_table[i][A].d) i-= 1;
        xmBihi = x-arctan_table[i][B].d;
        xmBilo = 0.0;
```

Listing 1: Code sample for arctan function from `crlibm` library

Vectorizable Reconstruction

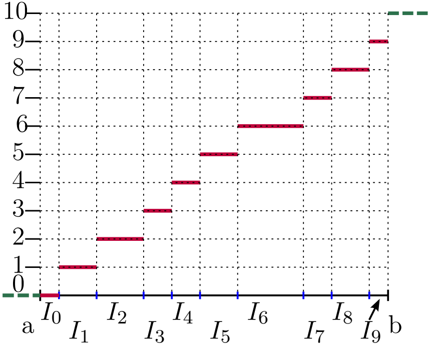
Splitting of
the domain $[a, b]$:



Vectorizable Reconstruction

To determine the subinterval we build a mapping function

$$P(x) = k, x \in I_k$$

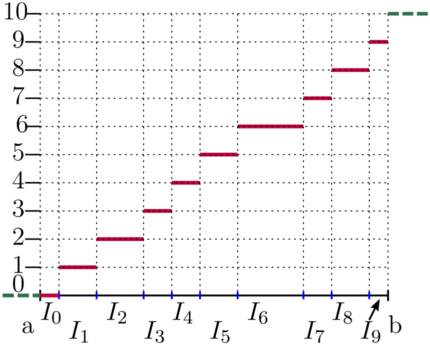


Vectorizable Reconstruction

To determine the subinterval we build a mapping function

$$P(x) = k, x \in I_k$$

$$p(x) : \lfloor p(x) \rfloor = k, x \in I_k$$



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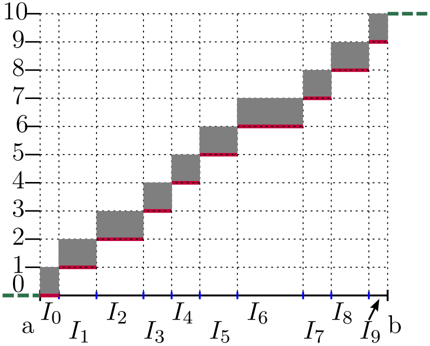
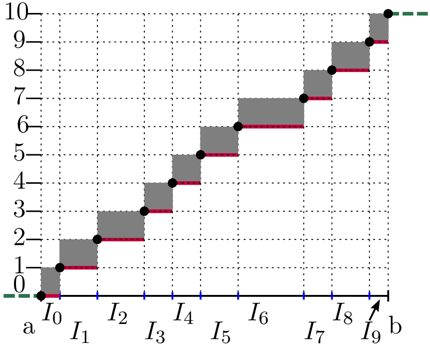


Vectorizable Reconstruction

Interpolation polynomial
with a posteriori condition check



Vectorizable Reconstruction

mapping function

$$P(x) = k, x \in I_k$$

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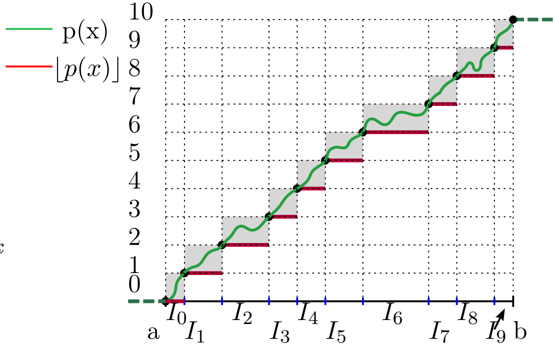
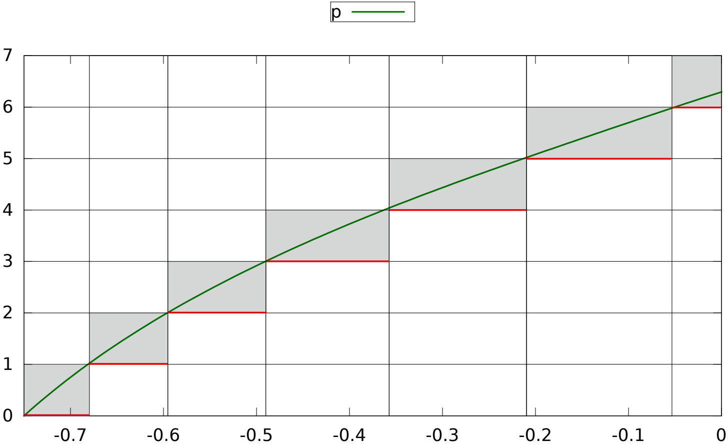


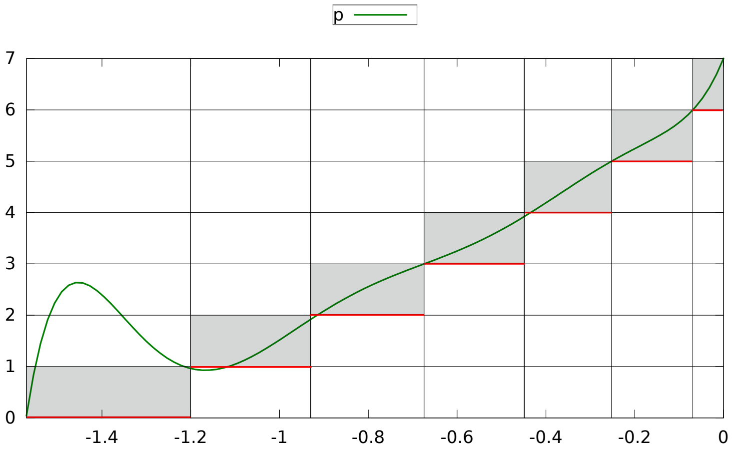
Example



f = asin(x); dom = [-0.75; 0];

$\bar{\varepsilon} = 2^{-48}$; $d_{\max} = 10$;

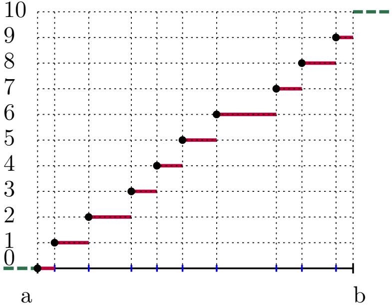
A Posteriori Condition Check Fails



$$f = \mathrm{atan}(x);\ \mathrm{dom} = [-\pi/2;\ 0];$$
$$\bar{\varepsilon} = 2^{-40};\ d_{\max} = 8;$$

How to Improve our Method

Interval Arithmetic approach:



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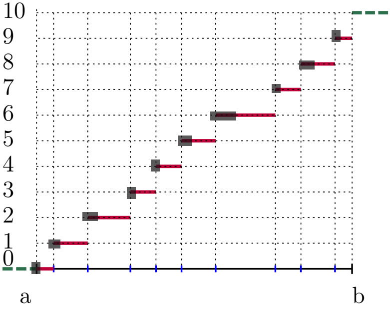

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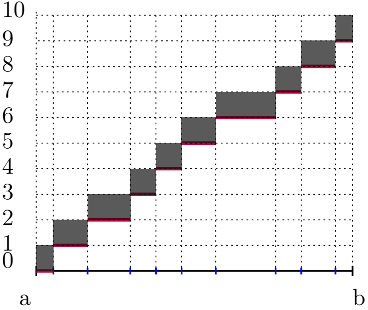

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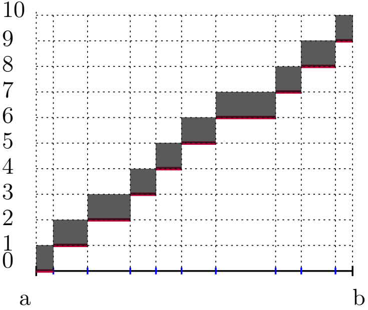

Interval system of linear algebraic equations

$$\begin{pmatrix} 1 & \mathbf{a}_0 & \cdots & \mathbf{a}_0^n \\ 1 & \mathbf{a}_1 & \cdots & \mathbf{a}_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \mathbf{a}_n & \cdots & \mathbf{a}_n^n \end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_n \end{pmatrix}$$

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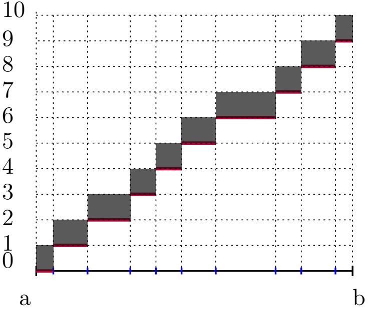
Interval Arithmetic approach:



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tolerance solution: $\Xi_{tol} = \left\{c \in \mathbb{R}^N \,|\, \forall a \in \mathbf{a}, \forall b \in \mathbf{b}, \ Ac = b\right\}$

united solution: $\Xi_{uni} = \left\{c \in \mathbb{R}^N \,|\, \exists a \in \mathbf{a}, \exists b \in \mathbf{b}, \ Ac = b\right\}$

S. Shary HdR thesis "Интервальные алгебраические задачи и их численное решение"

Achievements

- Generate the specified function versions in few minutes
- Detect algebraic properties automatically
- Use the improved (optimized) domain splitting algorithm
- Generation of the vectorizable implementations has started
- Extra bonuses: composite functions and a set of function variants

Code Generation for Mathematical Functions

Conclusion

Paving the road for mixed-radix arithmetic:

- Novel algorithm for radix conversion
- Algorithm for conversion from decimal character sequence to a binary FP number
- Worst cases search for FMA

Code generation for mathematical functions:

- Novel algorithm for domain splitting
- Novel approach to generate vectorizable implementations

Future Work

Mixed-Radix arithmetic

- Finish the worst cases search for FMA
- Start its implementation
- Research on other arithmetic operations
- Obtain test/comparison results for our scanf

Code generation of mathematical functions

- Filtering of special cases
- Improve vectorizable reconstruction:
  - Interval arithmetic for a priori approach
  - Overlapping intervals
  - Connection between reconstruction and domain splitting
- Add more parameters
- Integrate with N.Brunie's version

Perspectives

Long-term Future

- Integrate Metalibm to the glibc/gcc
- Apply the similar algorithms for filter generation
- Formal proof for scanf algorithm

Questions

- N. Brunie, F. de Dinechin, O. Kupriianova, and C. Lauter. Code generators for mathematical functions. In ARITH22 - June 2015, Lyon, France. Proceedings, pp 66-73. Best paper award.
- O. Kupriianova and C. Q. Lauter. A domain splitting algorithm for the mathematical functions code generator. In Asilomar Conference on Signals, Systems and Computers, Pacific Grove, CA, USA, November, 2014, pp 1271-1275.
- O. Kupriianova and C. Q. Lauter. Replacing branches by polynomials in vectorizable elementary functions. 16th GAMM-IMACS International Symposium on Scientific Computing, Computer Arithmetic and Validated Numerics, September 2014, Würzburg, Germany.
- O. Kupriianova and C. Q. Lauter. Metalibm: A mathematical functions code generator. In Mathematical Software - ICMS 2014 - 4th International Congress, Seoul, South Korea, August 2014. Proceedings, pp 713–717.
- O. Kupriianova, C. Q. Lauter, and J.-M. Muller. Radix conversion for IEEE754-2008 mixed radix floating-point arithmetic. In Asilomar Conference on Signals, Systems and Computers, Pacific Grove, CA, USA, November 2013, pp 1134–1138.