From Ordered Beliefs to Numbers: How to Elicit Numbers Without Asking for Them (Doable but Computationally Difficult)

Brian Cloteaux[1], Cristophe Eick[2],
Bernadette Bouchon-Meunier[3], and Vladik Kreinovich[1,3]

[1]Department of Computer Science
University of Texas at El Paso
El Paso, TX 79968, USA
email {cloteaux,vladik}@cs.utep.edu

[2]Department of Computer Science
University of Houston
Houston, TX 77204-3475, USA
email ceick@cs.uh.edu

[3]LAFORIA-IBP
Tour 46
University Paris VI
4, place Jussieu, 75252
Paris Cedex 05, France
email bouchon@laforia.ibp.fr

Abstract

One of the most important parts of designing an expert system is elicitation of the expert's knowledge. This knowledge usually consists of facts and rules. Eliciting these rules and facts is relatively easy, the more complicated task is assigning *weights* (numerical or interval-valued *degrees of belief*) to different statements from the knowledge base. Expert often cannot *quantify* their degrees of belief, but they can *order* them (by suggesting which statements are more reliable). It is, therefore, reasonable

to try to reconstruct the degrees of belief from such an ordering.

In this paper, we analyze when such a reconstruction is possible, whether it lead to unique values of degrees of belief, and how computationally complicated the corresponding reconstruction problem can be.

1 How to Elicit Numerical Degrees of Belief Without Asking for Them: Formulation of the Problem

1.1 It is necessary to describe degrees of belief

The core of a knowledge-based system is a body of knowledge elicited from the experts. This knowledge usually consists of facts and rules. Eliciting these rules and facts is a relatively easy task (to be more precise, it is doable; see, e.g., [8]). However, these facts and rules are not all we need. Experts usually have different degrees of belief in different statements: they have more belief in some of them and less belief in some others.

To describe these degrees of belief, expert systems usually use numbers from the interval $[0, 1]$, so that:

- 1 corresponds to the case when the expert is absolutely sure that the statement is true,
- 0 corresponds to the case when the expert is absolutely sure that the statement is false, and
- intermediate values describe intermediate degrees of belief.

To describe the expert's degree of belief in logical combinations of the original statements, such as $A_i \& A_j$, $A_i \vee A_j$, etc., we must generalize logical operations defined for two-valued logic to the case when truth values can take any values from an arbitrary interval $[0, 1]$. Many such generalizations are known [7, 10]. Generalizations of & are usually called *t-norms*, and generalizations of $\vee$ are called *t-conorms*.

1.2 For many experts, it is difficult to describe their degrees of belief

If an expert can provide us with the numbers that describe his degrees of belief, great. However, the very necessity of an expert system comes from the fact that experts often cannot quantify their knowledge. For example, the majority of people who consider themselves experts in driving a car cannot describe their driving in different situations in precise numerical terms, like "if the speed is 63 km/h, and there is a 40 km/h speed limit 1 mile ahead, step on the brakes for

1.3 sec". Instead, the expert can only say fuzzy statements like "if the lower speed limit is some distance ahead, step on the brakes for a little while".

If an expert cannot easily quantify the part of his knowledge for which there is, in principle, a natural quantification, then he is even less capable of quantifying his *degree of belief*, the quantity for which no natural quantification exists.

So, the problem is: *how to elicit the expert's degrees of belief without explicitly asking for them?*

1.2.1 This problem is easier for intelligent control systems, more difficult for expert systems

The problem of eliciting numbers without explicitly asking for numbers is somewhat easier for *intelligent control*, where in principle, we can relieve an expert from the necessity of answering any questions at all: we can simply:

- observe his behavior (either in the real control situation, or on a simulator), and
- extract his rules and degrees of belief from the observed control behavior (see, e.g., [1]).

This problem is much more complicated for *expert systems*, where such a solution is not available.

1.2.2 An additional complication: it is often more adequate to describe degrees of belief by intervals

One way to elicit the expert's degrees of belief is to ask an expert to estimate his degree of belief in a given statement by picking a number, say, on a scale from 0 to 10. This procedure is often used by the pollsters who simplify the respondent's task by adding the words like "absolutely sure", "sure", etc., to describe different numbers on the scale. If an expert picks, e.g., 6, then we take $6/10 = 0.6$ as the desired degree of belief.

Often, an expert is quite sure about his degree of belief, and this procedure works, but for some statements, he may be unsure about his degree of belief, and he may mark several values as equally well describing his degree of belief. For example, he may mark 5, 6, and 7. This means that this expert's degree of belief is best described not by a single number, but rather by an *interval* $[0.5, 0.7]$.

Even when a single number (say 6) is picked, it does not necessarily mean that the expert's degree of belief is exactly 0.6: it is quite possible that if we take a more detailed scale and ask an expert to mark his degree of belief on a scale of 0 to 100, he will mark not 60, but 62, or 58, or several numbers that are close to 60. The only thing that we can hope for is that the expert's choice for the 0 to 100 scale will be consistent with his choice for the 0 to 10 scale, i.e., that the resulting degree of belief from the 0 to 100 scale will be closer to 0.6 than to 0.5 or to 0.7. The borderlines between "closer to 0.6", "closer to 0.5", and "closer to 0.7", are the midpoints 0.55 and 0.65. As a result, from the fact that a person has marked 6 on a 0 to 10 scale, we can only conclude that his *actual* degree of belief is somewhere in the *interval* $[0.55, 0.65]$ (for a detailed description, see, e.g., [9]).

If we use intervals, then we must extend logical operations to the case of intervals. This extension is pretty straightforward:

- if the only thing we know about the degree of belief a_i in a statement A_i is that a_i belongs to an interval $[a_i^-, a_i^+]$, and
- if the only thing we know about the degree of belief a_j in a statement A_j is that a_j belongs to a certain interval $[a_j^-, a_j^+]$,
- then all possible values $a_i \& a_j$ for $a_i \in [a_i^-, a_i^+]$ and $a_j \in [a_j^-, a_j^+]$ (where & is the chosen t-norm) are possible values of degree of belief in $A_i \& A_j$,

Since a t-norm is assumed to be monotonically non-decreasing in both arguments, the smallest possible value of $a_i \& a_j$ is attained when both a_i and a_j attain the smallest possible values (i.e., when $a_i = a_i^-$ and $a_j = a_j^-$). Similarly, the largest possible values of $a_i \& a_j$ is attained when $a_i = a_i^+$ and $a_j = a_j^+$. Hence, the interval of possible values of $a_i \& a_j$ is

$$[a_i^-, a_i^+] \& [a_j^-, a_j^+] = [a_i^- \& a_j^-, a_i^+ \& a_j^+]. \tag{1}$$

Similarly, it is natural to define $\vee-$ and $\neg-$operations on interval values of degrees of belief as $[a_i^-, a_i^+] \vee [a_j^-, a_j^+] = [a_i^- \vee a_j^-, a_i^+ \vee a_j^+]$ and

$$\neg [a_i^-, a_i^+] = [\neg a_i^+, \neg a_i^-]. \tag{2}$$

Summarizing: In many cases, intervals are a more adequate description of degrees of belief. So, the problem of eliciting degrees of belief becomes: *how to elicit interval-valued degrees of belief?*

2 How to Elicit Numbers Without Asking for Them: Main Idea

2.1 Main idea: ordering instead of quantifying

It is often very difficult for an expert to *quantify* his degrees of belief, be it in the form of a number or in the form of an interval. However, an expert usually has no problem *ordering* his degrees of belief, i.e., deciding, for some pairs of statements A_i and A_j, that his belief in A_i is definitely not greater than his belief in A_j (we will denote this relation by $A_i \leq A_j$).

This relation can be easily reformulated in interval terms. Indeed, let us denote the interval of possible degrees of belief in A_i by $[a_i^-, a_i^+]$, and the interval of possible values of degree of belief in A_j by $[a_j^-, a_j^+]$. Then the relation $A_i \leq A_j$ means that for all $a_i \in [a_i^-, a_i^+]$ and for all $a_j \in [a_j^-, a_j^+]$, we have $a_i \leq a_j$. To guarantee this inequality for all possible values a_i and a_j, it is sufficient to make sure that this inequality holds for the largest possible a_i (i.e., for $a_i = a_i^+$) and for the smallest possible $a_j = a_j^-$. Hence, $A_i \leq A_j$ is equivalent to $a_i^+ \leq a_j^-$.

2.2 Modification of the main idea: we can also order logical combinations

In addition to ordering the degrees of belief in the original statements from the knowledge base, the experts can, as easily, order their degrees of belief in the *logical combinations* of these statements such as $A_i \& A_j$, $A_k \& \neg A_l$, etc.

This comparison makes the most sense for the most unambiguous operations such as "and" and "not". Operations "or" and "implies" are more ambiguous and sometimes even counter-intuitive:

- The implication of the type "if $2+2=5$, then witches like to wear black" is true in classical logic, but makes little intuitive sense.
- As for the natural-language "or":
  - sometimes it means logical (inclusive) "or", and
  - sometimes it means exclusive "or"; e.g., when I say "I seem to have seen John or Jack in the airport", I usually mean one of them and *not* both.

In view of this ambiguity, we would rather rely on statements that only use & and ¬.

Of these two connectives, & is less ambiguous, because in natural language, "not" also comes in two flavors:

- as a "weak" negation, as in "mature" as, strictly speaking, opposite to "young", but with a possibility to be still young to a certain extent; and
- as a "strong" negation, as in "not young any more", which is the exact opposite to "young".

So, preferably, we should only use &.

2.3 Main idea: final form

As a result, we arrive at the following idea:

- we have *preferences* $\leq$ between the & (and $\neg$) combinations of original statements, and
- we would like to determine the intervals of possible values that are *consistent* with these preferences in the sense that if $A_i \leq A_j$, then for the corresponding intervals $[a_i^-, a_i^+]$ and $[a_j^-, a_j^+]$, we have $a_i^+ \leq a_j^-$.

2.4 Reconstruction: in what sense?

If for each statement A_i, we have found an interval $[a_i^-, a_i^+]$ so that these intervals are consistent with the expert's preferences, then every sequence of narrower intervals $[{a'}_i^-, {a'}_i^+] \subseteq [a_i^-, a_i^+]$ will also be consistent with the same preferences. However, if we select these narrower intervals as the degrees of belief coming from the known preferences, then we will be kind of cheating, because by choosing narrower and less ambiguous intervals $[{a'}_i^-, {a'}_i^+]$ instead of the wider ones $[a_i^-, a_i^+]$ that are still consistent with the expert preferences, we are imposing certainty that is not in any way contained in the original preferences.

It is therefore desirable to choose, out of all possible combinations of intervals that are still consistent with the preferences, the combination in which the narrowest interval is the widest possible. In other words, if we have statements $A_1, \ldots, A_n$, then we would like to choose the intervals $[a_1^-, a_1^+], \ldots, [a_n^-, a_n^+]$ that are consistent with all preferences and for which the value

$$J(\vec{a}) = \min(a_1^+ - a_1^-, \ldots, a_n^+ - a_n^-) \tag{3}$$

is the largest possible.

Historical comment. This idea was first formulated (without mathematical formulas) in [3].

3 How to Elicit Numbers Without Asking for Them: Main Problems

In the previous section, we formulated the idea of how to elicit numerical degrees of belief without explicitly asking for them. With this idea, come the following problems:

- First of all, *is there* always *a solution* to this problem? Experts can give *inconsistent* preferences. Therefore, it is natural to ask: is it possible (and is it computationally easy) to check that a solution exists, i.e., that the expert's preferences are *consistent*?

- If the preferences are consistent, *does there exist an optimal assignment*?
- *Is* the resulting *optimal assignment* of interval degrees of belief *unique*, or for some preferences, there are several possible assignments with the same value of the maximized criterion $J(\vec{a})$?
- *How to* actually *compute* the optimal assignment? *Is the problem of computing* this assignment *computationally feasible*?

In this paper, we will consider these problems. In order to describe our solution to these problems, we will need two extra sections:

- First, we must formulate our idea in precise mathematical terms.
- Second, we must remind the reader what "computationally feasible" means.

4 Expert Preferences and How They Lead to Interval-Valued Degrees of Belief: Precise Definitions

Definition 1. (given information) *Let a positive integer n be given. This integer will be called the number of statements in a knowledge base.*

- *By a statement, we mean an expression of the type A_i, where $1 \le i \le n$.*
- *By an "and"-formula F, we mean either a statement, or an expression of the type $A_i \& \ldots \& A_j$, where $A_i, \ldots, A_j$ are different statements (i.e., A_i with different indices).*
- *By an "and"-"not" formula F, we mean either a statement, or an arbitrary expression that is obtained from statements by using $\&$ and $\neg$.*
- *By a preference, we mean an expression of the type $F \le G$, where F and G are formulas.*
- *By preference base P, we will mean a finite set of preferences.*

Definition 2. (desired assignment of interval-valued degrees of belief) *Let an integer n be given; let & be a t-norm [7, 10], and let $\neg$ be a negation operation (e.g., $\neg(a) = 1 - a$).*

- *By an* assignment of interval-valued degrees of belief *(or simply* assignment, *for short), we mean a tuple $\vec{a} = (a_1, \ldots, a_n)$ of n intervals $a_i = [a_i^-, a_i^+]$. The interval a_i is called the* degree of belief *in the statement A_i.*
- *For each assignment a and for each "and"-formula $F = A_i \& \ldots \& A_j$, we define the formula's degree of belief as the interval $f = [f^-, f^+]$, where $f^- = a_i^- \& \ldots \& a_j^-$ and $f^+ = a_i^+ \& \ldots \& a_j^+$.*
- *For each assignment a and for each "and"-"not" formula F, we define its degree of belief $f = [f^-, f^+]$ as the result of substituting intervals a_i instead of the statements A_i, and operations (1), (2) instead of & and $\neg$.*
- *We say that an assignment a is* consistent *with a preference $F \le G$ if the corresponding degrees of belief $f = [f^-, f^+]$ in F and $g = [g^-, g^+]$ in G satisfy the inequality $f^+ \le g^-$.*
- *We say that an assignment a is* consistent *with a preference base P if it is consistent with all preferences from this preference base.*
- *We say that a preference base is* consistent *if there exists an assignment a that is consistent with P.*
- *For every consistent preference base P, by an* optimal assignment, *we mean an assignment that is consistent with P and for which the value of the criterion $J(\vec{a})$ (formula (3)) is the largest among all assignments that are consistent with P.*

Comment. We want to know which computational reconstruction problems are feasible and which are intractable. Before we formulate our results, let us briefly recall how "feasible" and "intractable" are usually defined.

5 Feasible and Intractable: General Definitions (Brief Reminder)

5.1 Feasible

Some algorithms require lots of time to run. For example, some algorithms require the running time of $\ge 2^n$ computational steps on an input of size n. For reasonable sizes $n \approx 300$, the resulting running time exceeds the lifetime of the Universe and is, therefore, for all practical purposes, non-feasible.

In order to find out which algorithms are feasible and which are not, we must formalize what "feasible" means. This formalization problem has been studied in theoretical computer science; no completely satisfactory definition has yet been proposed.

The best known formalization is: an algorithm $\mathcal{U}$ is *feasible* iff it is *polynomial time*, i.e., iff there exists a polynomial P such that for every input x, the running time $t_{\mathcal{U}}(x)$ of the algorithm $\mathcal{U}$ on the input x is bounded by $P(|x|)$ (here, $|x|$ denotes the length of the input x).

This definition is not perfect, because there are algorithms that are polynomial time but that require billions of years to compute, and there are algorithms that require in a few cases exponential time but that are, in general, very practical. However, this is the best definition we have so far.

5.2 Intractable

For many mathematical problems, it is not yet known (1996) whether these problems can be solved in polynomial time or not. However, it is known that some combinatorial problems are as tough as possible, in the sense that if we can solve any of these problems in polynomial time, then, crudely speaking, we can solve many practically important combinatorial problems in polynomial time. The corresponding set of important combinatorial problems is usually denoted by NP, and problems whose fast solution leads to a fast solution of all problems from the class NP are called *NP-hard*. The majority of computer scientists believe that NP-hard problems are not feasible. For that reason, NP-hard problems are also called *intractable*. For formal definitions and detailed descriptions, see, e.g., [4].

6 Checking Consistency: Feasible for "and" Only, NP-Hard (For Some t-Norms) if We Use Both "and" and "not"

6.1 Feasible for "and" Only

It is easy to show that when we only use "and", then the preference base is always consistent:

PROPOSITION 1. *Every preference base that uses "and"-formulas is consistent.*

Proof. Take $a_i = [1, 1]$ for all i. It is easy to check that an arbitrary preference $F \le G$ between "and"-formulas is consistent with this assignment. Q.E.D.

6.2 NP-hard (for some t-norms) if we use both "and" and "not"

This result will not be so trivial, it will depend on the choice of a t-norm.

6.2.1 Types of t-norms: brief reminder

There are three main types of t-norms (see, e.g., [7, 10]):

- $a\&a' = \min(a, a')$;
- *Strictly Archimedean* t-norms. This class of t-norms is defined by some reasonable properties. A typical example is the "algebraic product" $a\&a' = a \cdot a'$. For our purposes, it is sufficient to know that there exists a complete classification of such operations: they are of the type $a\&a' = \varphi^{-1}(\varphi(a) \cdot \varphi(a'))$ for some strictly increasing continuous function φ (i.e., all such t-norms are in some sense isomorphic to the algebraic product).

These two classes are the ones most frequently used.

- *Non-strictly Archimedean* t-norms. This class of t-norms is also defined by some reasonable properties. A typical example is the "bold intersection" $a\&a' = \max(a + a' - 1, 0)$. For our purposes, it is sufficient to know that there exists a complete classification or such operations: they are of the type $a\&a' = \varphi^{-1}(\max(\varphi(a) + \varphi(a') - 1, 0))$ for some strictly increasing continuous function φ (i.e., all such t-norms are in some sense isomorphic to the bold intersection).

In this paper, we are interested in computational feasibility of different operations, so, we will assume that both the function φ and the inverse function φ^{-1} are feasible (i.e., can be computed in polynomial time).

6.2.2 Result

PROPOSITION 2.

- *For* $\& = \min$*, every preference base with "and"-"not" formulas is consistent.*
- *For every strictly Archimedean t-norm, the problem of checking consistency of preference bases with "and"-"not" formulas is NP-hard.*

Comments.

- For reader's convenience, the proof of this proposition, as well as the proofs of other results presented in this paper, are placed in a special (last) section of the paper.

- We have already mentioned that formulas that use "not" are more ambiguous that formulas that only use "and". Since it turns out that adding "not" also drastically increases the computational complexity of knowledge elicitation, in the remaining parts of the paper, we will primarily consider preference bases with preferences between "and" formulas,

7 For Consistent Preference Bases, Optimal Assignment Always Exists

PROPOSITION 3. *If a t-norm is a continuous function, and a preference base is consistent, then there exists an optimal assignment.*

8 Optimal Assignment is Not Always Unique

PROPOSITION 4.

- *For some consistent preference bases with "and" formulas, there exists exactly one optimal assignment.*
- *For some consistent preference bases with "and" formulas, there exist several optimal assignments.*

9 Computational Complexity of Computing the Optimal Assignment: Feasible for Some t-Norms, NP-Hard for Some Other t-Norms

9.1 NP-hard for some t-norms

PROPOSITION 5. *There exists a strictly Archimedean t-norm & for which the problem of computing the optimal assignment for preference bases with "and" formulas is NP-hard.*

9.2 Feasible for some other norms

We will show that this problem is feasible for the "bold intersection"

$$a \& a' = \max(a + a' - 1, 0).$$

Before we formulate the result, we must make the following remark:

- When an expert states a preference $F \le G$, where $F = A_i \& \ldots \& A_j$ and $G = A_k \& \ldots \& A_l$, i.e., that his belief in F is smaller than his degree of belief in G, then he implicitly assumes that his degree of belief in F is non-zero; otherwise, the inequality is trivially true and there is no reason to state it.

- For strictly Archimedean t-norms, if $a_i^+ > 0$, ..., and $a_j^+ > 0$, then $f^+ > 0$, and thus, the degree of belief in F is automatically different from $[0, 0]$.

- For the bold intersection, this is not automatically true, because we can have $a_i^+ = a_j^+ = 0.5$ and $a_i^+ \& a_j^+ = \max(0.5 + 0.5 - 1, 0) = 0$.

Hence, for the bold intersection, this condition needs to be specifically stated.

In other words, we need to modify the definition of consistency between the preference and an assignment:

Definition 2$'$. *We say that an assignment a is consistent with a preference $F \le G$ if the corresponding degrees of belief $f = [f^-, f^+]$ in F and $g = [g^-, g^+]$ in G satisfy the inequalities $[f^-, f^+] \ne [0, 0]$ and $f^+ \le g^-$.*

PROPOSITION 6. *For the "bold intersection" t-norm, there exists a feasible algorithm that computes, for every preference base with "and" formulas, the optimal assignment (in the sense of Definition 2$'$).*

10 Conclusions: Reconstructing Intervals from the Ordered Degrees of Belief is Doable but Computationally Difficult

In this paper, we addressed the following *problem*:

- for knowledge-based systems, it is necessary to quantify the expert's degrees of belief in their statements, but

- for many experts, it is very difficult to express their degrees of belief in numerical form.

To solve this problem, we proposed a new method of extracting the *numerical values* of degrees of belief from the expert's *ordering* of his degrees of belief.

The results of this paper lead to two conclusions:

- It is, in principle, *possible* to reconstruct numerical intervals of degrees of belief from the expert's order.

- For some t-norms, however, the corresponding reconstruction problem is computationally complicated; even, in the general case, intractable (NP-hard).

NP-hardness means that we cannot hope to design a feasible reconstruction algorithm that would work for all possible cases: we need *heuristic* methods.

Comment.

- The first result (that the reconstruction is possible) is clearly *positive*.
- The second one (that the reconstruction is computationally complicated) may seem negative.

However, we should not be discouraged by NP-hardness of the problem of finding the *initial* intervals of degrees of belief: Even if we get these degrees easily, the problem of computing the degrees of belief for different *queries* is also NP-hard [2], and hence, we need heuristic methods in knowledge-based systems anyway.

11 Proofs

11.1 Proof of Proposition 2

11.1.1 Case of $\& = \max$

In this case, we can easily show that every preference base is consistent: If we take $[a_i^-, a_i^+] = [0.5, 0.5]$ for all i, then $f = [f^-, f^+] = [0.5, 0.5]$ for all formulas F, and hence, $F \le G$ is always consistent with this assignment.

11.1.2 Case of a strictly Archimedean t-norm

Since all strictly Archimedean t-norms are isomorphic, it is sufficient to prove this result for the algebraic product $a\&a' = a \cdot a'$. Indeed, if for some preference base P, intervals $[a_i^-, a_i^+]$ form an assignment that is consistent with P for the algebraic product, then, for an arbitrary strictly Archimedean t-norm $a\&a' = \varphi^{-1}(\varphi(a)\cdot\varphi(a'))$, the assignment $[\varphi^{-1}(a_i^-), \varphi^{-1}(a_i^+)]$ is consistent with the same preference base P for this t-norm.

Due to this remark, it is sufficient to show that the problem of checking consistency is NP-hard for the algebraic product.

We will prove this result by showing that if we can solve this consistency problem in polynomial time, then we will be able to solve the so called *propositional satisfiability* problem 3-SAT (we will describe it in a minute) in polynomial time. Since 3-SAT is known to be NP-hard (it is actually historically the first problem proven to be NP-hard), we can thus conclude that our consistency-checking problem is also NP-hard: Indeed:

- The fact that 3-SAT is NP-hard means that whenever if can solve this problem is polynomial time, then we can solve an arbitrary NP-problem in polynomial time.

- If we can solve the consistency-checking problem in polynomial time, then we will be able to solve also 3-SAT in polynomial time, and thus, solve all the problems from the class NP in polynomial time. Hence, our problem is NP-hard.

Let us now describe the satisfiability problem.

- Let $x_1, \ldots, x_m$ be a finite list of *Boolean* (*propositional*) variables, i.e., variables that take two possible values: "true" and "false".
- By a *literal* $a, \ldots$, we mean either a variable x_i, or its negation.
- By a *disjunction* D, we mean an expression of the type $a \vee b \vee c$, where a, b, and c are literals.
- By a 3-*SAT* formula, we mean an expression of the type $D_1 \& \ldots \& D_k$, where $D_1, \ldots, D_k$ are disjunctions.
- By a *Boolean vector*, we mean a sequence of n truth values $x_1, \ldots, x_n$. For each Boolean vector, we can define the truth value of a 3-SAT formula F by substituting the values x_i into the formula F.
- By a 3-*SAT problem*, we mean the following problem:
  - *Given:* a 3-SAT formula;
  - *To compute:* a Boolean vector that makes it true, or a message "no satisfying vector" if no such vector exists.

Let us now describe the reduction of 3-SAT formulas to the problem of checking consistency of reference bases.

Let F be an arbitrary 3-SAT formula with variables $x_1, \ldots, x_m$. The corresponding reference base will describe $n = m + 9$ statements $A_1, \ldots, A_m, A_{m+1}, \ldots, A_{m+9}$ and consist of the following preferences:

- For each statement A_{m+i}, $1 \le i \le 9$, two special preferences:

$$A_{m+1} \le \neg A_{m+1} \ \mathit{and} \ \neg A_{m+1} \le A_{m+1}.$$

- For each statement A_i, $1 \le i \le m$, a preference

$$A_i \& \neg A_i \le A_{m+1} \& A_{m+2} \& A_{m+3} \& A_{m+4}.$$

- For each disjunction $D_j = a \vee b \vee c$, a preference

$$A_{m+1} \& A_{m+2} \& \ldots \& A_{m+9} \le A \& B \& C,$$

  where:

- for a positive literal $a = x_i$, A means A_i, and
- for a negative literal $a = \neg x_i$, A means $\neg A_i$.

Let us show that this reference base is consistent if and only if the original 3-SAT formula is satisfiable.

1. If the original 3-SAT formula F is satisfiable, i.e., if there exists a Boolean vector x_i (with values from the set $\{0,1\}$) that makes F true, then we can take $a_i = [x_i, x_i]$ for $i \le m$, and $a_{m+i} = [0.5, 0.5]$.

For this assignment:

- $a^+_{m+i} = 0.5 \le 1 - a^+_{m+i} = 0.5$, and similarly, the second preference related to A_{m+i} is also true.
- Since each statement A_i, $1 \le i \le m$, gets assigned the value "true" or "false" (0 or 1), the truth value of $A_i \& \neg A_i$ will be 0, and therefore, the assignment is consistent with the preference relations special for these variables (of the type $A_i \& \neg A_i \le \ldots$)
- Since the Boolean vector $x_1, \ldots, x_m$ makes the formula $F = D_1 \& \ldots \& D_k$ true, it means (according to the properties of "and" in 2-valued logic) that it makes all the disjunctions D_j true. Hence, the truth value of $A \& B \& C$ will be 1 ($[1,1]$), and therefore, the preference corresponding to this disjunction is also consistent with this assignment.

So, if the formula F is satisfiable, then this preference base is consistent.

2. Let us now show that if the preference base is consistent, then the original formula F is satisfiable.

Indeed, let $\vec{a} = ([a^-_1, a^+_1], \ldots)$ be the assignment that is consistent with the above-defined preference base P. Let us use this assignment to construct a satisfying vector for the formula F.

2.1. From the fact that $\vec{a}$ is consistent with the preferences $A_{m+i} \le \neg A_{m+i}$ and $\neg A_{m+i} \le A_{m+i}$, we conclude that $a^+_{m+i} \le 1 - a^+_{m+i}$ and that $1 - a^-_{m+i} \le a^-_{m+i}$. From the first inequality, by moving the term $-a^+_{m+i}$ to the left-hand side, we conclude that $2a^+_{m+i} \le 1$ and that $a^+_{m+1} \le 0.5$. Similarly, from the second inequality, we conclude that $a^-_{m+1} \ge 0.5$. Hence, $0.5 \le a^-_{m+i} \le a^+_{m+i} \le 0.5$, and therefore, none of these four inequalities can be a strict inequality, they must all be equalities, i.e., $a^-_{m+i} = a^+_{m+i} = 0.5$.

2.2. From the fact that $\vec{a}$ is consistent with the preference $A_i \& \neg A_i \le A_{m+1} \& \ldots \& A_{m+4}$, we conclude that $a^+_i(1 - a^-_i) \le a^-_{m+1} \cdot \ldots \cdot a^-_{m+4}$. Since we already know that $a^-_{m+i} = 0.5 = 1/2$, we conclude that $a^+_i(1 - a^-_i) \le 1/16$.

- Since $a^-_i \le a^+_i$, we conclude that $a^-_i(1 - a^-_i) \le a^+_i(1 - a^-_i) \le 1/16$.
- Similarly, from the fact that $1 - a^+_i \le 1 - a^-_i$, we conclude that $a^+_i(1 - a^+_i) \le a^+_i(1 - a^-_i) \le 1/16$.

So, we have an inequality $z(1-z) \le 1/16$ for both $z = a_i^-$ and for $z + a_i^+$.

The function $z(1-z) = z - z^2$ is strictly increasing for $z \le 0.5$, and strictly decreasing for $z \ge 0.5$. For $z = 1/8$ and $z = 7/8$, we have

$$z(1-z) = (1/8) \cdot (7/8) > 1/16.$$

Therefore, from $z(1-z) \le 1/16$, we can conclude that either $z < 1/8$, or $z > 7/8$. Hence, both a_i^- and a_i^+ are either $< 1/8$, or $> 7/8$. Since $a_i^- \le a_i^+$, we can, in principle, have three different cases:

1) $a_i^+ < 1/8$; in this case, $a_i^- \le a_i^+ < 1/8$, and $a_i = [a_i^-, a_i^+] \subseteq [0, 1/8)$.

2) $a_i^- > 7/8$; in this case, $7/8 < a_i^- \le a_i^+$, and $a_i = [a_i^-, a_i^+] \subseteq (7/8, 1]$.

3) $a_i^- < 1/8$ and $a_i^+ > 7/8$.

Let us show that the third case is impossible. Indeed, in this case, $a_i^+ > 7/8$ and $1 - a_i^- > 7/8$, hence, $a_i^+(1 - a_i^-) > (7/8)^2 > 1/16$, which contradicts to our conclusion that $a_i^+(1 - a_i^-) \le 1/16$.

Hence, we are left with only two cases: 1) and 2). Now, we are ready to define the Boolean values that will satisfy our formula F:

1) In the first case, when $[a_i^-, a_i^+] \subseteq [0, 1/8)$, we take $x_i = 0$.

2) In the second case, when $[a_i^-, a_i^+] \subseteq (7/8, 1]$, we take $x_i = 1$.

2.3. Let us show that the Boolean vector chosen in part 2.2 of this proof satisfies the formula F, i.e., makes all its disjunctions D_j true.

Let us first start with the remark about literals. If a literal a is the variable, then, as we have proven, either $[a^-, a^+] \subseteq [0, 1/8)$ and $x_i =$ "false", or $[a^-, a^+] \subseteq (7/8, 1]$ and $x_i =$ "true". If a literal a is a negation of a variable, the similar inclusions can be easily deduced. Hence, for every literal a, we have one of the two cases:

- $[a^-, a^+] \subseteq [0, 1/8)$; in this case, the literal a is "false" for the chosen Boolean vector x_i.
- $[a^-, a^+] \subseteq (7/8, 1]$; in this case, the literal a is "true" for the chosen Boolean vector x_i.

For each disjunction, from the fact that the assignment $\vec{a}$ is consistent with the preference

$$A_{m+1} \& \ldots \& A_{m+9} \le A \& B \& C,$$

we can conclude that $a_{m+1}^+ \cdot \ldots \cdot a_{m+9}^+ \le a^- \cdot b^- \cdot c^-$. We know from part 2.1 of this proof that $a_{m+i}^+ = 0.5$, hence, $a^- \cdot b^- \cdot c^- \ge (1/2)^9$.

If all three literals a, b, and c were false, we would have $a^- < 1/8$, $b^- < 1/8$, and $c^- \le 1/8$, hence, $a^- \cdot b^- \cdot c^- < (1/8)^3 = (1/2)^9$, which contradicts to the inequality that we have just derived. Hence, the assumption that all three literals are false is wrong, so, one of these literals must be true. Hence, the disjunction D_j is true.

So, all disjunction are true, i.e., the formula F is indeed satisfied.

3. So, a 3-SAT formula F is satisfiable if and only if the corresponding preference base is consistent. As we have already mentioned in the beginning of the proof, this means that checking consistency is NP-hard.

Comment. In this proof, we assumed that $\neg a = 1 - a$. For different $\neg$−operations, instead of 0.5, we must take the truth value a_0 for which $a_0 = \neg(a_0)$. Q.E.D.

11.2 Proof of Proposition 3

Each assignment $\vec{a}$ consists of $2n$ numbers from the interval $[0, 1]$. Thus, the set of all possible assignments is a subset of a unit cube $[0, 1]^{2n}$ in the $(2n)$−dimensional space and therefore, it is a bounded set in a $(2n)$−dimensional space R^{2n}.

To prove that the optimal assignment always exists, we will show:

- first, that the optimality criterion $J(\vec{a})$ (defined by the formula (3)) is a continuous function, and
- that the set C of all assignments that are consistent with a given preference base P is a closed and bounded (hence, compact) subset of R^{2n}.

Then, the existence of the optimal assignment will follow from the well-known result from calculus that every continuous function on a compact set attains its maximum.

- The function $J(\vec{a})$ is clearly a *continuous* function of all its $2n$ arguments a_i^- and a_i^+.
- The set C of all assignments that are consistent with a given preference base P is a subset of the bounded set of all assignments, and is, therefore, itself *bounded.*
- Let us show that this set C is *closed*, i.e., that if $\vec{a}^{(N)} \in C$ for all N, and $\vec{a}^{(N)} \to \vec{a}$, then $\vec{a} \in C$. In other words, we want to prove that if for every N, assignments $a^{(N)}$ are consistent with all the preferences from the preference base P, then the limit assignment $\vec{a}$ is consistent with the same preferences.

  Since the t-norm & is continuous, for every "and"-formula F, the expressions for $f^-(\vec{a})$ and $f^+(\vec{a})$ that correspond to an assignment $\vec{a}$ are continuous functions of the assignment values a_i^- and a_i^+. Thus, from $\vec{a}^{(N)} \to \vec{a}$, we can conclude that $f^+(\vec{a}^{(N)}) \to f^+(\vec{a})$ and $g^-(\vec{a}^{(N)}) \to g^-(\vec{a})$.

Let $F \leq G$ be a preference from the preference base P. The fact that each of the assignments $a^{(N)}$, $N \geq N_0$, is consistent with each preference $F \leq G$ means that we have $f^+(\vec{a}^{(N)}) \leq g^-(\vec{a}^{(N)})$ for the degrees of belief f^+ and g^- that correspond to the assignment $\vec{a}^{(N)}$.

Hence, from $f^+(\vec{a}^{(N)}) \leq g^-(\vec{a}^{(N)})$, we conclude, in the limit, that $f^+(\vec{a}) \leq g^-(\vec{a})$, i.e., that the assignment $\vec{a}$ is also consistent with the preference $F \leq G$.

So, the limit assignment $\vec{a}$ is consistent with each of the preferences. Therefore, the limit assignment is consistent with the preference base. Thus, the set C is closed.

Closeness of C was the only thing missing in the proof. With closeness proven, the proposition is now proved as well. Q.E.D.

11.3 Proof of Proposition 4

11.3.1 Example of uniqueness

Let us take $n = 2$, and the preference base that consists of only one preference $A_1 \leq A_2$.

The assignment is consistent with this preference iff $a_1^+ \leq a_2^-$. In other words, the two intervals $[a_1^-, a_1^+]$ and $[a_2^-, a_2^+]$ that form the assignment can intersect at most at one point. Hence, the total width of these two intervals cannot exceed the total width of the interval $[0, 1]$ on which they both lie. In other words, the sum of the widths is ≤ 1. Hence, at least one of these widths is ≤ 0.5 (otherwise, the sum of their widths would exceed 1). Thus, the smallest of these two widths is ≤ 0.5. In other words, for each assignment $\vec{a}$ that is consistent with this preference base, the value of the optimality criterion (3) cannot exceed 0.5.

It is easy to find the assignment for which this criterion is equal to exactly 0.5: $a_1 = [0, 0.5]$ and $a_2 = [0.5, 1]$.

Let us show that this is the only possible assignment with this value of the optimality criterion. Indeed, let us assume that $\vec{a}$ is an assignment for which the value of the criterion (3) is equal to 0.5. This means that the widths of both intervals $[a_i^-, a_i^+]$ are 0.5 or greater. Since the sum of these widths cannot exceed 1, none of these widths can exceed 0,5, so, each of these interval has a width of exactly 0.5. The total width of the part of $[0, 1]$ that is not covered by these two intervals is thus $1 - 0.5 - 0.5 = 0$. Hence, the entire interval $[0, 1]$ is covered; in particular, the point 0 is covered. Since $a_2^- \geq a_1^+ = a_1^- + 0.5 \geq 0.5 > 0$, the point 0 cannot be covered by the second interval. It is, therefore, covered by the first interval. So, $a_1^- = 0$, hence $a_1^+ = a_1^- + 0.5 = 0 + 0.5 = 0.5$, and the only possible location of the second interval is $a_2 = [0.5, 1]$.

So, for this particular preference base, there is exactly one optimal assignment.

11.3.2 Example of non-uniqueness

Let us take $n = 3$ and the preference base that consists of the only preference $A_1 \leq A_2$. Then, similar to the proof for the uniqueness example, we can show that the optimal value of the criterion (3) is 0.5, and that this value is attained when $a_1 = [0, 0.5]$ and $a_2 = [0.5, 1]$. However, we no longer have uniqueness, because for the third interval a_3, the only restriction that stems from optimality is that its width be ≥ 0.5. There are many intervals with this property, e.g., $[\alpha, \alpha + 0.5]$ for any $\alpha \in [0, 0.5]$. So, for this preference base, there exist several different optimal assignments. The proposition is proven.

Comment. It may seem, at first glance, that the non-uniqueness is caused by the fact that one of the statements in the original knowledge base is not restrained by any preference relation. However, a similar example can be proposed in which every statement is restrained by some preference: e.g., $n = 5$, and P consists of the following three preferences: $A_1 \leq A_2$, $A_2 \leq A_3$, and $A_4 \leq A_5$. Then, for the optimal assignment, we will similarly conclude that the optimal value of the criterion (3) is $1/3$, and that this value is attained when $a_1 = [0, 1/3]$, $a_2 = [1/3, 2/3]$, and $a_3 = [2/3, 1]$. However, optimal assignment is not unique, because the only conditions on a_4 and a_5 are that their widths are $\geq 1/3$ and that a_5 follows a_4. For example, for every $\alpha \in [0, 1/3]$, we can take $a_4 = [\alpha, \alpha + 1/3]$ and $a_5 = [\alpha + 1/3, \alpha + 2/3]$.

11.4 Proof of Proposition 5

1. Let us show that for some strictly Archimedean t-norm the problem of finding the optimal assignment is NP-hard. Namely, we will show that this problem can be reduced to the propositional satisfiability for 3-SAT formulas.

Let F be a propositional formula with variables $x_1, \ldots, x_m$. Let us form a preference base P that describes $2m + 5$ statements $A_1, \ldots, A_m, A_{-1}, \ldots, A_{-m}$, B_1, B_1', B_2, B_3, B_4, and consists of the following preferences:

- $B_1 \leq B_2 \leq B_3 \leq B_4$; $B_1' \leq B_2$.
- For every $i \leq m$, three sets of preferences: $B_1 \leq A_i \leq B_4$, $B_1 \leq A_{-i} \leq B_4$, and $B_1 \& B_2 \leq A_i \& A_{-i} \leq B_2 \& B_3$.
- For every disjunction $D_j = a \vee b \vee c$, a preference $B_1 \& B_1' \& B_2 \leq A \& B \& C$, where:
  - for positive literals $a = x_i$, we take $A = A_i$, and
  - for negative literals $a = \neg x_i$, we take $A = A_{-i}$.

The general t-norm can be describe as $a \& a' = \varphi^{-1}(\varphi(a) \cdot \varphi(a'))$. In other words, for every t-norm, there exists a strictly increasing function $\varphi : [0, 1] \to$

$[0,1]$ for which $\varphi(a\&a') = \varphi(a)\cdot\varphi(a')$. For our proof, we will need another mapping that maps $\varphi(a) \in [0,1]$ into $-\log(\varphi(a)) \in [0,\infty]$ and transforms multiplication into addition: $\psi(a\&a') = \psi(a) + \psi(a')$. The function $\psi(a) = -\log(\varphi(a))$ is a strictly decreasing function from $[0,1]$ to $[0,\infty]$.

2. The consistency between each preference and an assignment can be expressed in terms of this function ψ. Namely:

- A preference $A_i\&\ldots\&A_j \le A_k\&\ldots\&A_l$ is equivalent to $a_i^+\&\ldots\&a_j^+ \le a_k^-\&\ldots\&a_l^-$.
- Since ψ is a decreasing function, this inequality, in its turn, is equivalent to $\psi(a_k^-\&\ldots\&a_l^-) \le \psi(a_i^+\&\ldots\&a_j^+)$.
- Finally, due to our choice of ψ, this is equivalent to $\psi(a_k^-)+\ldots+\psi(a_l^-) \le \psi(a_i^+)+\ldots+\psi(a_j^+)$.

3. Let us first show that if a formula F is satisfiable, then this preference base has an assignment in which the width of each interval is ≥ 0.25 (i.e., an assignment with $J(\vec{a}) \ge 0.25$). Indeed, let x_i be the Boolean vector that makes the formula F true. Then, we take:

- $b_1 = b_1' = [0, 0.25]$, $b_2 = [0.25, 0.5]$, $b_3 = [0.5, 0.75]$, $b_4 = [0.75, 1]$;
- $a_i = [0.5, 0.75]$ and $a_{-i} = [0.25, 0.5]$ for those i for which $x_i = 1$, and
- $a_i = [0.25, 0.5]$ and $a_{-i} = [0.5, 0.75]$ for those i for which $x_i = 0$.

It is easy to check that this assignment is consistent with all the above preferences.

4. Let us now show that if some assignment $\vec{a}$ is consistent with the above preferences, and $J(\vec{a}) \ge 0.25$, then, from this assignment, we can extract a satisfying vector for the formula F.

4.1. First, from $B_1 \le B_2 \le B_3 \le B_4$, we can conclude that the intervals $b_1,\ldots,b_4$ follow one another practically without intersections (at most, the neighboring intervals may intersect in one point). The width of each interval is ≥ 0.25, therefore, their total width is ≥ 1, but since they are all in the interval $[0,1]$, it cannot be greater than 1. So, each of these four intervals must be exactly of width 0.25, and they must fill the entire interval $[0,1]$. Thus, $b_1 = [0, 0.25]$, $b_2 = [0.25, 0.5]$, $b_3 = [0.5, 0.75]$, and $b_4 = [0.75, 1]$.

Since a similar argument can be repeated with b_1' instead of b_1, we can conclude that $b_1' = b_1 = [0, 0.25]$.

4.2. From $B_1 \le A_i \le B_4$, we conclude that $0.25 \le a_i^- \le a_i^+ \le 0.75$.

Since the width of each interval is ≥ 0.25, we have $a_i^- \le a_i^+ - 0.25 \le 0.75 - 0.25 = 0.5$. Hence, for every i, we have $0.25 \le a_i^- \le 0.5$. Similarly, we have $0.25 \le a_{-i}^- \le 0.5$.

For the further proof, let us denote the difference $a_i^- - 0.25$ by α, and the difference $0.5 - a_{-i}^-$ by β. Then, we have $a_i^- = 0.25 + \alpha$ and $a_{-i}^- = 0.5 - \beta$.
4.3. From $B_1 \& B_2 \le A_i \& A_{-i}$, we conclude that

$$\psi(0.25 + \alpha) + \psi(0.5 - \beta) \le \psi(0.25) + \psi(0.5). \tag{4}$$

Similarly, from $A_i \& A_{-i} \le B_2 \& B_3$, we conclude that $a_i^+ \& a_{-i}^+ \le b_2^- \& b_3^- = 0.5 \& 0.75$. Since the width of each interval is ≥ 0.25, we have $0.5 + \alpha = a_i^- + 0.25 \le a_i^+$, and $0.75 - \beta = a_{-i}^- + 0.25 \le a_{-i}^+$. Hence, from the monotonicity of a t-norm, we conclude that $(0.5 + \alpha) \& (0.75 - \beta) \le 0.5 \& 0.75$. In terms of ψ, this inequality takes the form

$$\psi(0.5) + \psi(0.75) \le \psi(0.5 + \alpha) + \psi(0.75 - \beta). \tag{5}$$

Let us show that for an appropriate ψ, these two inequalities can only be true for $\alpha = \beta = 0$ or for $\alpha = \beta = 0.25$.

To show that, we will take a function $\psi(x)$ that is strictly convex for $x \in [0.25, 0.5]$, and strictly concave for $x \in [0.5, 0.75]$. In other words, we will assume that:

- the second derivative is negative ($\psi''(x) < 0$) for $x \in [0.25, 0.5)$; and
- the second derivative is positive ($\psi''(x) > 0$) for $x \in (0.5, 0.75]$.

As a result, the derivative $\psi'(x)$ strictly decreases for $x \in [0.25, 0.5]$, and strictly increases for $x \in [0.5, 0.75]$.
4.4. Let us consider the function $\psi(0.25+\lambda)+\psi(0.5-\lambda)$, where $0 \le \lambda < 0.125$. The derivative of this function w.r.t. λ is equal to $\psi'(0.25+\lambda)-\psi'(0.5-\lambda)$. For $\lambda < 0.125$, we have $0.25+\lambda < 0.5-\lambda$. Both values $0.25+\lambda$ and $0.5-\lambda$ belong to the interval $[0.25, 0.5]$ on which the derivative $\psi'(x)$ is strictly decreasing. Therefore, $\psi'(0.25 + \lambda) - \psi'(0.5 - \lambda) > 0$.

The derivative of the function $\psi(0.25 + \lambda) + \psi(0.5 - \lambda)$ is positive, and therefore, this function is strictly increasing for $\lambda \in [0, 0.125]$. Hence, comparing the value of this function for an arbitrary $\lambda > 0$ and the value of the same function for $\lambda = 0$, we get the following inequality:

$$\psi(0.25 + \lambda) + \psi(0.5 - \lambda) > \psi(0.25) + \psi(0.5). \tag{6}$$

We have proven this inequality for $\lambda \le 0.125$. If $0.125 \le \lambda < 0.25$, then we can take $\lambda' = 0.25 - \lambda$. For λ', we have $\lambda' < 0.125$ and hence,

$$\psi(0.25 + \lambda') + \psi(0.5 - \lambda') > \psi(0.25) + \psi(0.5). \tag{7}$$

But, by definition of λ', $0.5 - \lambda' = 0.25 + \lambda$, and $0.25 + \lambda' = 0.5 - \lambda$, hence, we get the same inequality (6).

So, (6) is proven for all $\lambda \in (0, 0.25)$.

4.5. Let us prove that if $\beta \neq 0$ and $\beta \neq 0.25$, then $\alpha > \beta$. We will prove this statement by reduction to a contradiction.

Let $\alpha \leq \beta \neq 0$. Since ψ is a decreasing function, from $\beta \geq \alpha$, we can conclude that $\psi(0.25+\beta) \leq \psi(0.25+\alpha)$. From (4), we can now conclude that

$$\psi(0.25+\beta) + \psi(0.5-\beta) \leq \psi(0.25) + \psi(0.5),$$

which contradicts to (6).

4.6. Similarly, by considering the function ψ on the interval $[0.5, 0.75]$, we can conclude that if $\beta \neq 0$ and $\beta \neq 0.25$, then $\alpha < \beta$.

Since it is impossible to have $\alpha > \beta$ and $\alpha < \beta$ at the same time, the only remaining possibility is $\beta = 0$ or $\beta = 0.25$. Similarly, we can conclude that $\alpha = 0$ or $\alpha = 0.25$.

From the inequalities (4) and (5) it now follows that we cannot have $\alpha = 0$ and $\beta = 0.25$, and we cannot have $\alpha = 0.25$ and $\beta = 0$. Hence, we have either $\alpha = \beta = 0$, or $\alpha = \beta = 0.25$, i.e.,

- either $a_i^- = 0.25$ and $a_{-i}^- = 0.5$,
- or $a_i^- = 0.5$ and $a_{-i}^- = 0.25$.

4.7. Let us now define the values of the Boolean variables accordingly:

- If $a_i^- = 0.25$ and $a_{-i}^- = 0.5$, we take $x_i = 0$.
- If $a_i^- = 0.5$ and $a_{-i}^- = 0.25$, we take $x_i = 1$.

Now, from the condition $B_1 \& B_1' \& B_2 \leq A \& B \& C$, we conclude that $0.25 \& 0.25 \& 0.5 \leq a^- \& b^- \& c^-$, and therefore, that it is impossible for all three literals to be false. Thus, for every disjunction, at least one of the literals is true; hence, the disjunction itself is true, and so, the formula F is satisfied by the Boolean variables $x_1, \ldots, x_n$.

To complete the proof, we must give an example of a decreasing function $\psi(x) : [0,1] \to [0,\infty]$ that is convex ($\psi''(x) < 0$) for $x \in [0.25, 0.5]$ and concave ($\psi''(x) > 0$) for $x \in [0.5, 0.75]$. Then, we will be able to define $\varphi(x) = \exp(-\psi(x))$, and the desired t-norm as $a \& a' = \varphi^{-1}(\varphi(a) \cdot \varphi(a'))$. The following is an example of such a function $\psi(x)$:

- $\psi(x) = 3/x$ for $x \leq 0.25$;
- $\psi(x) = 0.5 + 0.25 \cdot \sin(2\pi x)$ for $0.25 \leq x \leq 0.75$;
- $\psi(x) = 1 - x$ for $x \geq 0.75$.

The theorem is proven.

11.5 Proof of Proposition 6

The algorithm is easy to describe:

- First, we design a linear programming problem based on the given preference base. This problem will have $2n+1$ variables: a_i^- $(1 \le i \le n)$, a_i^+ $(1 \le i \le n)$, and J, the optimization criterion $J \to \max$, and the following linear inequalities:
  - For every $i = 1, \ldots, n$, the following four inequalities:

    $$0 \le a_i^-, \; a_i^- \le a_i^+, \; a_i^+ \le a_i^- + J, \; and \; a_i^+ \le 1.$$

  - For every preference of the type $A_i \& \ldots \& A_j \le A_k \& \ldots \& A_l$, two inequalities

    $$a_i^+ + \ldots + a_j^+ - N_l + 1 > 0, \tag{8}$$

    $$a_i^+ + \ldots + a_j^+ - N_l \le a_k^- + \ldots + a_l^- - N_r, \tag{9}$$

    where N_l and N_r denote the number of statements correspondingly in the left-hand side and in the right-hand side of the preference $F \le G$.
- Then, we apply one of the known polynomial-time algorithms for solving linear programming problems (see, e.g., [6, 5]) to the resulting problem.

To prove that this algorithm is correct, let us first show that the above inequalities (8) and (9) are indeed equivalent to the consistency between the assignment $\vec{a}$ and the preference $F \le G$. Indeed, this consistency can be expressed as $a_i^+ \& \ldots \& a_j^+ \ne 0$ and

$$a_i^+ \& \ldots \& a_j^+ \le a_k^- \& \ldots \& a_l^-.$$

Since & is the bold intersection, we can transform these inequalities into the following formulas:

$$\max(a_i^+ + \ldots + a_j^+ - N_l + 1, 0) > 0 \tag{10}$$

$$\max(a_i^+ + \ldots + a_j^+ - N_l + 1, 0) \le \max(a_k^- + \ldots + a_l^- - N_r + 1, 0). \tag{11}$$

The inequality (10) has the form $\max(z, 0) > 0$ for some expression z. This inequality cannot be true for $z \le 0$ and it is always true for $z > 0$, so (10) is equivalent to (8).

The left-hand side of the inequality (11) is positive and hence, the right-hand side is positive as well. The only way for $\max(z, 0)$ to be positive is when $z > 0$, then $\max(z, 0) = z$. So, the expression $\max(z, 0)$ in the right hand side can be replaced by z. If we do a similar replacement for the left-hand side, we end up with an inequality

$$a_i^+ + \ldots + a_j^+ - N_l + 1 \le a_k^- + \ldots + a_l^- - N_r + 1.$$

If we subtract 1 from both sides of this inequality, we get the desired inequality (9).

So, the linear inequalities describe the fact that the intervals $[a_i^-, a_i^+]$ are consistent with preferences. For given intervals, the conditions on J say that J cannot exceed the width of each interval. The largest possible value with this property is the width of the narrowest interval, i.e., exactly the formula (3). The proposition is proven.

Acknowledgments. This work was partially supported by NASA Grant No. NAG 9-757.

References

[1] B. Bouchon-Meunier and V. Kreinovich, "Simulating Fuzzy Control as a New Method of Eliciting Membership Functions", *Proceedings of the International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems (IPMU'96)*, Granada, Spain, July 1–5, 1996 (to appear).

[2] L. S. Chee, "Computing the Value of a Boolean Expression with Interval Inputs is NP-Hard", *Reliable Computing* (to appear).

[3] C. F. Eick and N. M. Mehta, "Decision blocks – a tool for rule-based decision making involving imperfect knowledge", *IEEE Transactions on Systems, Man, and Cybernetics*, June 1993.

[4] M. R. Garey and D. S. Johnson, *Computers and intractability: a guide to the theory of NP-completeness*, Freeman, San Francisco, 1979.

[5] N. Karmarkar, "A new polynomial-time algorithm for linear programming", *Combinatorica*, 1984, Vol. 4, pp. 373–396.

[6] L. G. Khachiyan, "A polynomial-time algorithm for linear programming", *Soviet Math. Dokl.*, 1979, Vol. 20, No. 1, pp. 191–194.

[7] G. Klir and B. Yuan, *Fuzzy sets and fuzzy logic: theory and applications*, Prentice Hall, Upper Saddle River, NJ, 1995.

[8] X. Ménage and B. Bouchon-Meunier, "Competitive learning of possibility distributions", *Proceedings of the Third IEEE Conference on Fuzzy Systems*, IEEE Press, 1994, Vol. I, pp. 442–445.

[9] H. T. Nguyen and V. Kreinovich, "Towards theoretical foundations of soft computing applications", *International Journal on Uncertainty, Fuzziness, and Knowledge-Based Systems*, 1995, Vol. 3, No. 3, pp. 341–373.

[10] H. T. Nguyen and E. A. Walker, *A First Course in Fuzzy Logic*, CRC Press, Boca Raton, Florida, 1996 (to appear).