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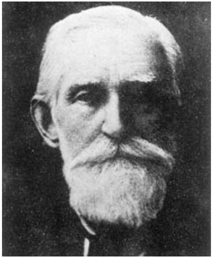
Chebychev Approximation by Polynomials, Rational Functions and Splines



Pafnuti Lwowitsch Chebychev (1821 - 1894)

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Best Approximation and Minimal deviation

Let $(F, \|\cdot\|)$ be a normed space and V a non-empty subset of F.
For given $f \in F$, determine $v^* \in V$, such that

$$\|f - v^*\| \leq \|f - v\| \text{ for all } v \in V.$$

$\boldsymbol{v^*}$ is the **best approximation** of f w.r.t. V

$\boldsymbol{\varrho_V(f)} = \|f - v^*\| \; (= \inf_{v \in V} \|f - v\|)$: **Minimal deviation**

Main Questions

Main Questions and Problems (depending on F, $\|\cdot\|$ and V):

1. Existence
2. Uniqueness
3. Characterization
4. (Numerical) construction, algorithms
5. Lower and upper bounds for $\varrho_V(f)$

Definitions and Notations

In this talk: $F = C[a,b]$

In the first (and main) part: V is a Haar subspace[1] of $C[a,b]$.[2]

Definition

An m-dimensional subspace V of $C[a,b]$ is called **Haar subspace**, if each $v \in V$, $v \neq 0$, has at most $m-1$ zeros in $[a,b]$.

[1] Alfred Haar, 1885 - 1933

[2] Sometimes also Chebychev subspace

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Important examples

- Π_n, the space of polynomials of degree at most n, is a $(n + 1)$-dimensional Haar subspace on each interval $[a, b]$.
- T_n, the space of trigonometric polynomials of degree at most n, is a $(2n + 1)$-dimensional Haar subspace on $[0, 2\pi)$.

[1] Alfred Haar, 1885 - 1933

[2] Sometimes also Chebychev subspace

L_1-Approximation

L_1-Approximation I

Let, for all $h \in C[a,b]$,

$$\|h\|_1 = \int_a^b |h(x)|dx$$

denote the L_1-**Norm**.

Theorem (Jackson 1930)

Let $f \in C[a,b]$ and let V be a Haar subspace of $C[a,b]$. Then f possesses one and only one best L_1-approximation w.r.t. V.

L_1-Approximation II

Theorem (Kripke and Rivlin 1965)

Let $f \in C[a,b]$ and let V be a Haar subspace of $C[a,b]$. Then $v^* \in V$ is the best L_1-approximation of f, if and only if

$$\int_a^b v(x) \cdot \operatorname{sgn}(f - v^*)(x)dx \leq \int_Z |v(x)|dx, \qquad (1)$$

for all $v \in V$, where Z denotes the set of zeros of $(f - v^*)$.
If, in addition, $(f - v^*)$ has only finitely many zeros, then (1) simply reads

$$\int_a^b v(x) \cdot \operatorname{sgn}(f - v^*)(x)dx = 0.$$

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Chebychev Approximation

Chebychev norm

The norm

$$\|h\|_\infty = \max_{x \in [a,b]} |h(x)|$$

is called

- maximum norm
- uniform norm
- L_∞-norm
- Chebychev norm

Chebychev norm

The norm

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- maximum norm
- uniform norm
- L_∞-norm
- Chebychev norm

From now on throughout the talk!

Negative example (for uniquenes)

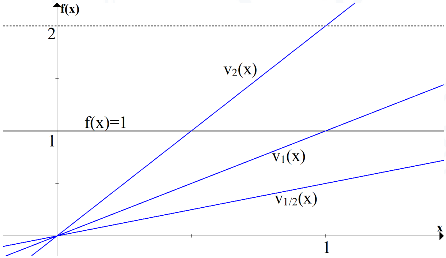
Let $[a, b] = [0, 1]$, $V = \operatorname{span}\{x\}$ and $f(x) \equiv 1$.

Since $v(0) = 0$ for all $v \in V$, we have $\varrho_V(f) \geq 1$.

Set $v_\alpha(x) = \alpha x$ with $0 \leq \alpha \leq 2$. Then

$$\|f - v_\alpha\| = \max_{x \in [0,1]} |(1 - \alpha x)| = 1.$$

So v_α is best approximation for all $0 \leq \alpha \leq 2$.



Haar subspaces are the „right“ spaces for Chebychev approximation:[3]

Theorem (Haar/Kolmogorov 1918/48)

Let V be an n-dimensional subspace of $C[a, b]$.

1. If V is a Haar subspace, then each $f \in C[a, b]$ possesses one and only one best approximation w.r.t. V.
2. If there exists a function $v \in V$, $v \neq 0$, with at least n zeros in $[a, b]$, then there exists an $\overline{f} \in C[a, b]$, which has more than one best approximation w.r.t. V.

[3] Andrej Nikolajevich Kolmogorov, 1903 - 1987

Let $h \in C[a,b]$, and let $E(h)$ denote the set of **extremal points** of h:

$$E(h) = \{x \in [a,b]; |h(x)| = \|h\|_\infty\}.$$

Definition (Alternant)

Let $h \in C[a,b]$ and $m \in \mathbb{N}$. A set of points

$$a \leq x_1 < x_2 < \cdots\cdots < x_m \leq b$$

is called an **alternant** of h of length m, if

$$\{x_1, x_2, \ldots, x_m\} \subset E(h)$$

and

$$h(x_i) = -h(x_{i+1}) \text{ for } i = 1, 2, \ldots m-1.$$

Chebychev Approximation

Characterization

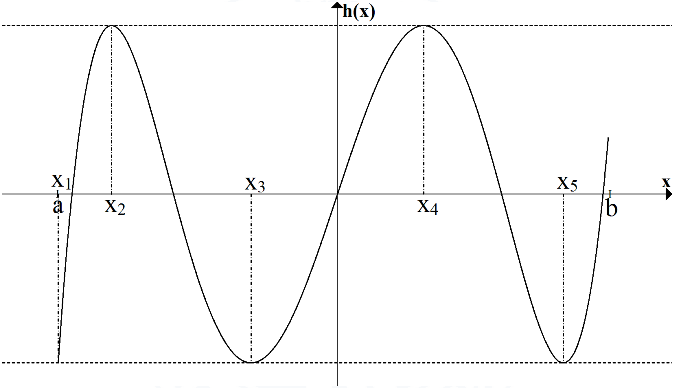


Figure: Alternant of length 5

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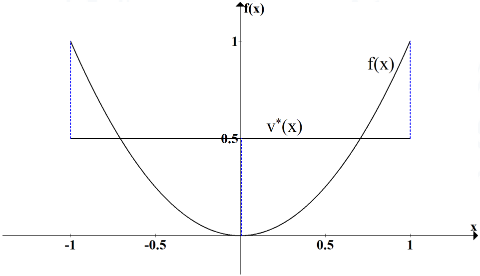
Fundamental Theorem (Chebychev 1859)

Let V be an n-dimensional Haar subspace of $C[a,b]$ and $f \in C[a,b]$.
Then $v^* \in V$ is the best approximation of f w.r.t. V if and only if the **error function** $(f - v^*)$ has an alternant of length $n+1$.

Example: $f(x) = x^2$ on $[-1,1]$, Π_1 ($\mapsto n = 2, n+1 = 3$)



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Characterization

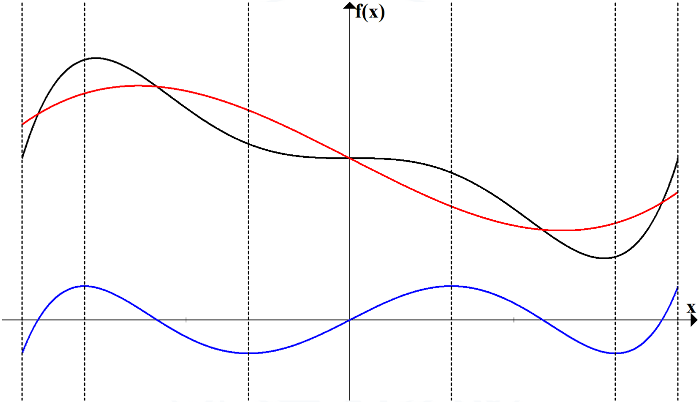

Figure: Black: Function $f(x)$. Red: Best Approximation $v^*(x)$. Blue: Error function $(f - v^*)(x)$

Chebychev's Theorem is fundamental for the whole theory. Among others, it leads to

- estimates (inclusion) of the minimal deviation (de la Vallée Poussin[4])
- iterative algorithm for the construction of the best approximation (Remez[5])

[4]Charles Jean Gustave Nicolas Baron de la Vallée Poussin, 1866 – 1962
[5]Evgenii Yakovlevich Remez, 1895 – 1975

De la Valleé Poussin

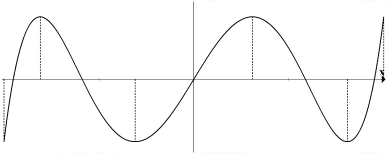

Figure: Alternant of error function

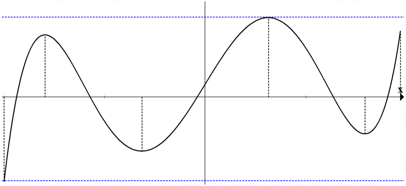

Figure: Almost Alternant of error function

Theorem (de la Valleé Poussin)

Let V be an n-dimensional Haar subspace of $C[a,b]$ and $f \in C[a,b]$. Assume that there exists a function $\tilde{v} \in V$ and $n+1$ points

$$a \leq t_1 < t_2 < \cdots\cdots < t_{n+1} \leq b$$

such that

$$(f - \tilde{v})(t_i) \neq 0 \text{ for } i = 1, 2, \ldots, n+1$$

and

$$\operatorname{sgn}\{(f - \tilde{v})(t_i)\} = - \operatorname{sgn}\{(f - \tilde{v})(t_{i+1})\} \text{ for } i = 1, 2, \ldots, n.$$

Then

$$\min_{1 \leq i \leq n+1} |(f - \tilde{v})(t_i)| \leq \varrho_V(f) \leq \|(f - \tilde{v})\|.$$

De la Valleé Poussin

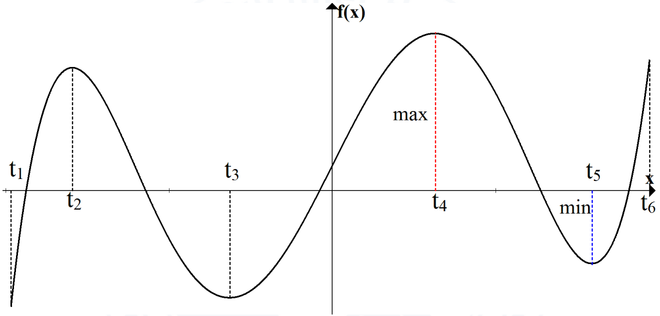

Figure: Illustration of de la Valleé Poussin

De la Valleé Poussin

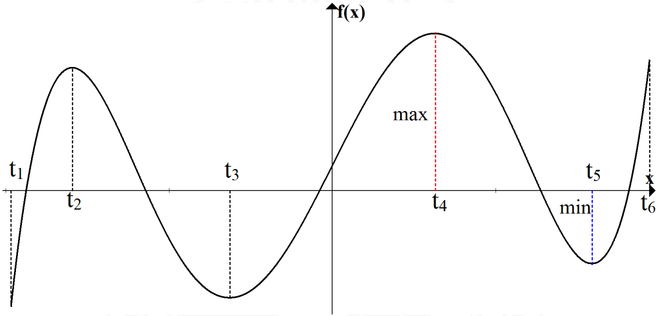

Figure: Illustration of de la Valleé Poussin

Leads to the fundamental **Remez algorithm**.

Basic idea: Iterative construction of

- a sequence of functions $v^k(x)$ with $\lim\limits_{k\to\infty} v^k(x) = v^*(x)$, and simultaneously
- a sequence of numbers λ^k with $\lim\limits_{k\to\infty} |\lambda^k| = \varrho_V(f)$

Remez algorithm

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Fundamental lemma

Let V be an n-dimensional Haar subspace of $C[a,b]$ and $f \in C[a,b]$. Then, for an arbitrary point set (a so-called **reference**)

$$R = \{a \leq x_1 < x_2 < \cdots < x_{n+1} \leq b\}$$

there exists a uniquely determined function $v \in V$ and a uniquely determined real number λ, such that

$$(f - v)(x_i) = (-1)^i \cdot \lambda \quad \text{for } i = 1, 2, \ldots, n+1.$$

Chebychev Approximation

Remez algorithm

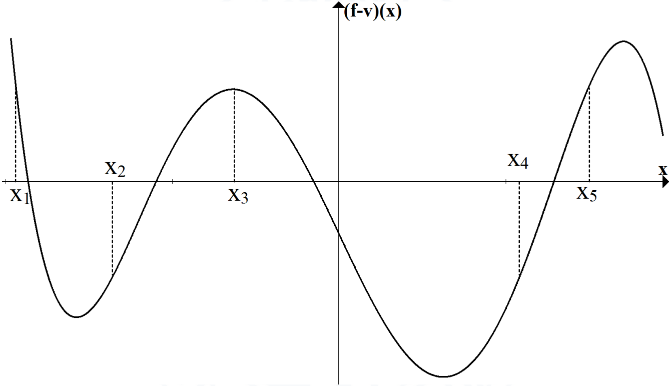

Figure: Illustration of the fundamental lemma: $n = 4$, $\lambda =$ length of dotted lines

Remez algorithm

- Step 0: Choose an arbitrary reference R^0 on $[a, b]$. Set $k = 0$.

Remez algorithm

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- Step 1:Determine $v^k(x)$ and λ^k according to the fundamental lemma.
  - If $\|f - v^k\| = |\lambda^k|$, then $v^k(x)$ is the best approximation and R^k is the alternant. Stop algorithm and praise the Lord.
  - If $\|f - v^k\| > \lambda^k$, but $(\|f - v^k\| - \lambda^k) < \varepsilon$, stop. Otherwise proceed with Step 2.

Remez algorithm

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- Step 2: Determine an extremal point ξ^k of $(f - v^k)$. Assume $x_i^k < \xi^k < x_{i+1}^k$ (cyclic). Replace x_i^k or x_{i+1}^k by ξ^k s.t. the alternating behaviour of the error function remains valid. This defines a new reference R^{k+1}

Remez algorithm

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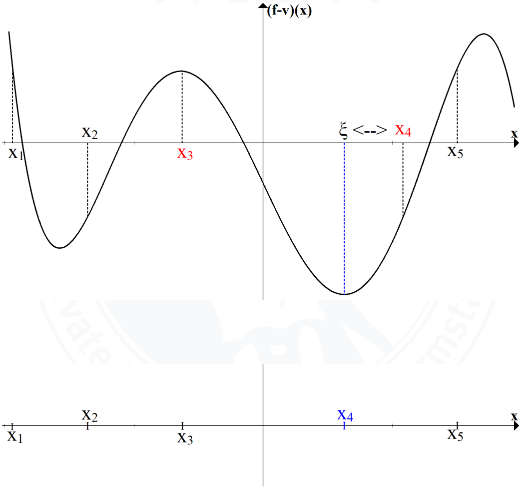
- Step 3: Set $k := k+1$ and proceed with Step 1.

Remez algorithm



Main Theorem (Remez)

With the notations and definitions from above, the following statements hold:

1. For all $k \in \mathbb{N}_0$:
$$|\lambda^k| < |\lambda^{k+1}| \leq \varrho_V(f)$$
2. $\lim_{k\to\infty} |\lambda^k| = \varrho_V(f)$
3. $\lim_{k\to\infty} \|v^k - v^*\| = 0.$

Remez algorithm: Numerical Example

Example (due to H.J.Oberle): Approximate

$$f(x) = \frac{\sin(x)}{x} \quad \text{on} \quad \left[0, \frac{\pi}{2}\right]$$

by even polynomials of degree 6, i.e.

$$p(x) = a_0 + a_1 x^2 + a_2 x^4 + a_3 x^6.$$

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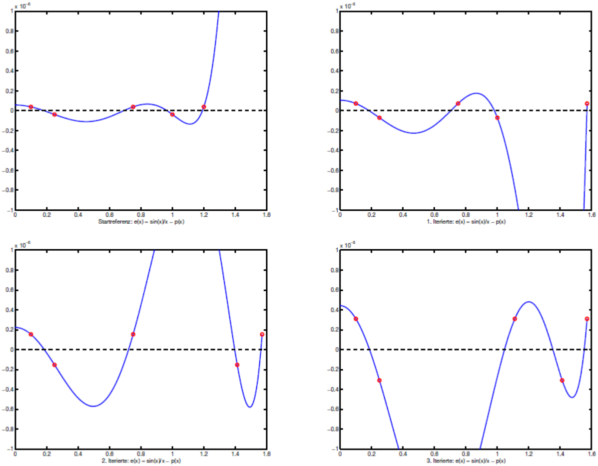
Remez algorithm: Numerical Example



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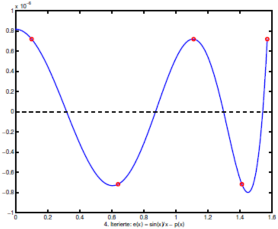
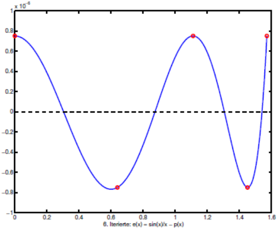
Remez algorithm: Numerical Example





$$p^*(x) = 0.999999 - 0.166667x^2 + 0.008313x^4 - 0.000185x^6$$

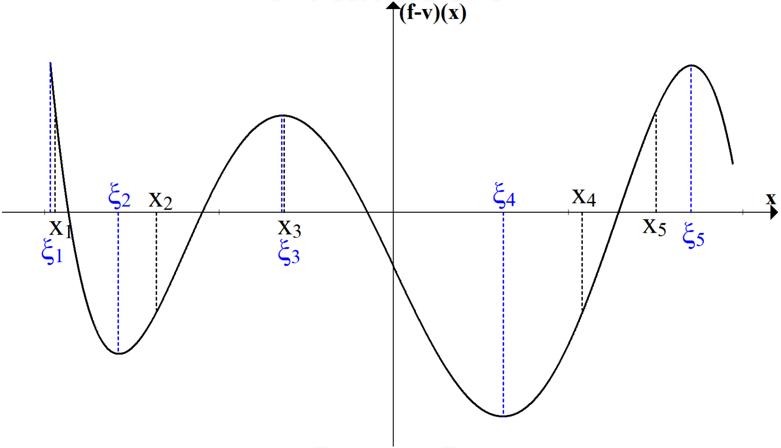
$$\varrho_V(f) = 0.75439 \cdot 10^{-6}$$

Remez algorithm with simultaneous exchange

Remark: Convergence of remez algorithm is **linear** in general.

Remez algorithm with **simultaneous exchange** of all reference points leads to **quadratic convergence** (Theorem of Veidinger 1960).

Remez algorithm with simultaneous exchange



Weak Haar Spaces

Weak Haar Space

Definition (Weak Haar Space)

An m-dimensional subspace V of $C[a, b]$ is called **Weak Haar subspace**, if each $v \in V$ has at most $m - 1$ sign changes in $[a, b]$.

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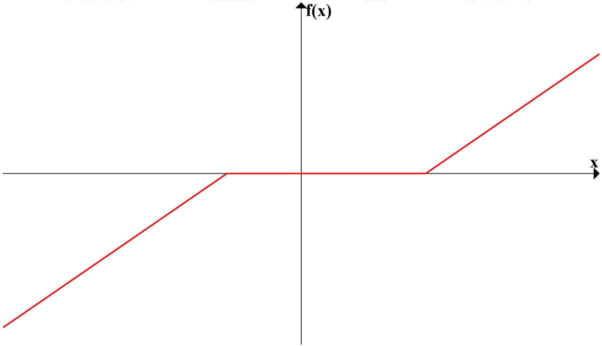


Weak Haar Space

Theorem (Jones and Karlowitz 1970)

Let V be an n-dimensional Weak Haar subspace of $C[a, b]$ and $f \in C[a, b]$.

1. Then there exists a best approximation $v^* \in V$ of f w.r.t. V, such that the error function $(f - v^*)$ has an alternant of length $n + 1$.
2. If, for some $v^* \in V$, the error function $(f - v^*)$ has an alternant of length $n + 1$, then v^* is a best approximation of f.

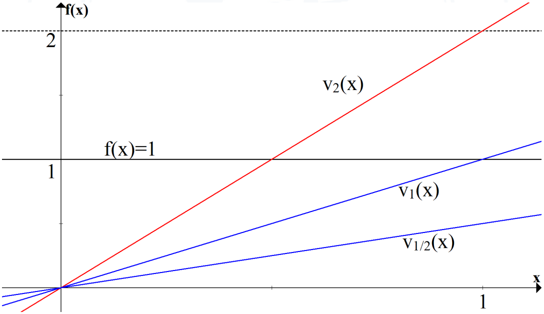
Remark: Uniqueness is lost!

Weak Haar Space

Example: Let $[a,b] = [0,1]$, $V = \text{span}\{x\}$ and $f(x) \equiv 1$.

Set $v_\alpha(x) = \alpha x$ with $0 \le \alpha \le 2$. Then

- v_α is best approximation for all $0 \le \alpha \le 2$.
- $(f - v_2)$ has an alternant of length 2.



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Definition (Spline)

With $k, m \in \mathbb{N}, m \geq 2$, let be given a set of $k+1$ fixed **knots**

$$a = t_0 < t_1 < \cdots\cdots < t_{k-1} < t_k = b$$

on $[a, b]$. A function s is called **spline of order** m (w.r.t. these knots), if

1. $s \in C^{m-2}[a, b]$
2. the restriction of s to each interval $[t_i, t_{i+1}]$ is a polynomial of degree $m-1$.

Example: $m = 2$

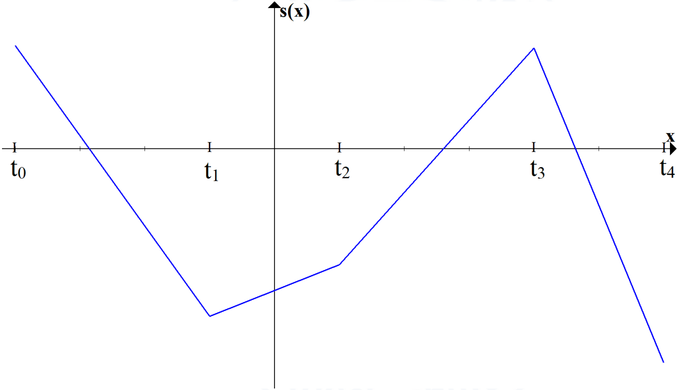

Figure: Linear spline

Splines

Example: $m = 3$

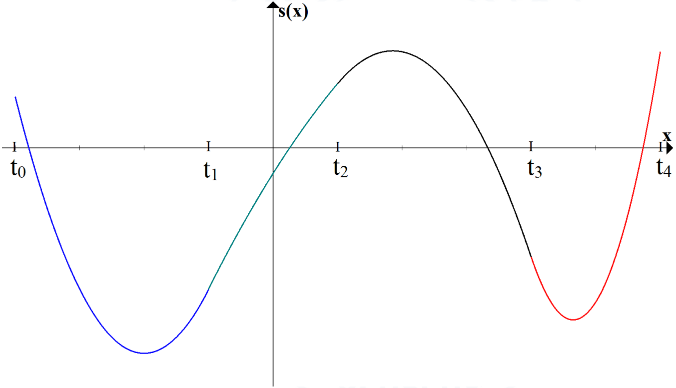

Figure: Quadratic spline

Theorem

Let

$$S_m(t_0, \ldots, t_k)$$

denote the set of all splines of order m w.r.t. the above defined knot set.
Then $S_m(t_0, \ldots, t_k)$ is a Weak Haar subspace of $C[a, b]$ of dimension

$$\dim S_m(t_0, \ldots, t_k) = m + k - 1.$$

Splines: Characterization

For best spline approximations, there is a characterization theorem by Rice and Schumaker.[6]

Theorem (Rice and Schumaker 1968)

Let $f \in C[a,b]$. Then a spline $s^* \in S_m(t_0, \ldots, t_k)$ is a best approximation of f, if and only if there is a knot interval $[t_p, t_{p+q}]$, such that the error function $(f - s^*)(x)$ has an alternant of length $m + q$ on $[t_p, t_{p+q}]$.

[6] John R. Rice, born 1934; Larry L. Schumaker, born 1939

Uniqueness is very complicated; the following result is due to Nürnberger.[7]

Theorem (Nürnberger 1978)

Let $f \in C[a,b]$. Then a spline $s^* \in S_m(t_0, \dots, t_k)$ is the (strongly) unique best approximation of f, if and only if the following statements on the error function $(f - s^*)(x)$ are true:

[7] Günther Nürnberger, 1948 - 2013

Splines: Strong uniqueness

Uniqueness is very complicated; the following result is due to Nürnberger.[7]

Theorem (Nürnberger 1978)

Let $f \in C[a,b]$. Then a spline $s^* \in S_m(t_0, \dots, t_k)$ is the (strongly) unique best approximation of f, if and only if the following statements on the error function $(f - s^*)(x)$ are true:

1. $(f - s^*)(x)$ has an alternant of length $m + k$ on $[a, b]$.
2. For each (relevant) j and ν, $(f - s^*)(x)$ has an alternant of length $j + 1$ on the intervals

$$[t_0, t_j),\ (t_{n-j}, t_n],\ \ (t_\nu, t_{\nu+m+j-1}).$$

[7] Günther Nürnberger, 1948 - 2013

There is also an **algorithm** for computing best spline approximations, due to Sommer[8] and Nürnberger (1983).

Exchange Algorithm of **Remez type**, but much more complicated.

Two fundamental differences:

[8] Manfred Sommer, born 1950

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Exchange Algorithm of **Remez type**, but much more complicated.

Two fundamental differences:

1. System of equations for computing new reference might be unsolvable $\rightarrow$ modified exchange rule

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Exchange Algorithm of **Remez type**, but much more complicated.

Two fundamental differences:

1. System of equations for computing new reference might be unsolvable $\to$ modified exchange rule
2. Spline approximations are not unique in general $\to$ convergence (proof) very complex

[8] Manfred Sommer, born 1950

Nonlinear Approximation

Chebychev Approximation

Nonlinear Case

So far: Linear approximation
Now: Nonlinear approximation (e.g.: rational functions, free knot splines)

Definition (Tangent Space)

Let $F \subset C[a,b]$ be a set of functions of the form $f(p,x)$, depending on a parameter vector $p = (p_1, p_2, \ldots, p_k)$. The **tangent space** of F at the point $f(p,x)$ is the linear space

$$T(p) = \operatorname{span}\left\{\frac{\partial}{\partial p_1} f(p,x), \frac{\partial}{\partial p_2} f(p,x), \ldots, \frac{\partial}{\partial p_k} f(p,x)\right\}$$

Set $d(p) = \dim(T(p))$.

Definition (Haar condition)

The set F satisfies the

- **local Haar condition**, if for every parameter vector p the tangent space $T(p)$ is a Haar subspace

Definition (Haar condition)

The set F satisfies the

- **local Haar condition**, if for every parameter vector p the tangent space $T(p)$ is a Haar subspace
- **global Haar condition**, if for every choice of parameter vectors p and q, $p \neq q$, the difference function

$$f(p,x) - f(q,x)$$

has at most $d(p) - 1$ zeros on $[a,b]$.

Semi-uniqueness and characterization.[9]

Theorem (Meinardus 1961)

Let $f \in C[a,b]$. Suppose that F satisfies the local and the global Haar condition. Then the following holds:

1. f has at most one best approximation w.r.t. F.

[9] Günter Meinardus, 1926 - 2007

Semi-uniqueness and characterization.[9]

Theorem (Meinardus 1961)

Let $f \in C[a,b]$. Suppose that F satisfies the local and the global Haar condition. Then the following holds:

1. f has at most one best approximation w.r.t. F.
2. The function $v^* \in F$ is the best approximation of f, if and only if the error function $(f - v^*)(x)$ has an alternant of length $d(p) + 1$.

[9] Günter Meinardus, 1926 - 2007

Special case: **Rational functions**

$$f(p,x) = \frac{P_m(x)}{Q_n(x)} = \frac{a_0 + a_1 x + \cdots + a_m x^m}{1 + b_1 x + \cdots + b_n x^n}$$

i.e.

$$p = (a_0, a_1, \ldots, a_m, b_1, b_2, \ldots b_n)$$

Tangent Space:

$$\text{span}\left\{ \frac{1}{Q_n(x)}, \frac{x}{Q_n(x)}, \ldots, \frac{x^m}{Q_n(x)}, \frac{-xP_m(x)}{Q_n^2(x)}, \ldots, \frac{-x^n P_m(x)}{Q_n^2(x)} \right\}$$

Satisfies the local and global Haar condition!

Uniqueness and characterization:

Theorem (Chebychev)

Let $f \in C[a,b]$ and $R_{m,n}$ the set of rational functions as above. Then the following holds:

1. f has one and only one best approximation w.r.t. $R_{m,n}$.

Uniqueness and characterization:

Theorem (Chebychev)

Let $f \in C[a,b]$ and $R_{m,n}$ the set of rational functions as above. Then the following holds:

1. f has one and only one best approximation w.r.t. $R_{m,n}$.
2. The function $r^* \in R_{m,n}$ is the best approximation of f, if and only if the error function $(f - r^*)(x)$ has an alternant of length $l + 1$, where

$$l = \begin{cases} m + n + 1 - \min(m - m_1, n - n_1), & \text{if } P_m(x) \not\equiv 0 \\ m + 1, & \text{if } P_m(x) \equiv 0 \end{cases}$$

and m_1, n_1 denotes the exact degrees of P_m and Q_n resp.

Free Knot Splines

So far: Splines with fixed knots. **Now**: Free knots.

Definition (Free Knot Splines)

A function $s : [a, b] \mapsto \mathbb{R}$ is called **spline of order** m **with** $k-1$ **free knots**, if there exists a set of knots

$$a = t_0 \leq t_1 \leq \cdots\cdots \leq t_{k-1} \leq t_k = b$$

s.t. $s \in S_m(t_0, \ldots, t_k)$.

The set $S_{m,k}$ of all these splines is (highly) nonlinear, since the knots are parameters!

The following result on the tangent space goes in principle back to Wulbert:[10]

Theorem (Wulbert 1973)

The tangent space of the spline set $S_{m,k}$ at the point s is the space of splines with the same knots, but (explicitly known) higher multiplicities.

So the tangent space is a weak Haar space!

Idea: Approximate it by Haar spaces.

[10] Daniel Wulbert, born 1940

Definition (Gauß transform)

For $h \in C(\mathbb{R})$ and $t > 0$, the mapping resp. the function

$$G(h,t)(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} h(u) \cdot \exp\left(-\frac{(u-x)^2}{4t}\right) du$$

is called **Gauß transform** of h with parameter t (if it exists).

Theorem

1. For all $x \in \mathbb{R}$:

$$\lim_{t \to 0} G(h,t)(x) = h(x).$$

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2. If

$$\operatorname{span}\{h_1(x), h_2(x), \ldots, h_m(x)\}$$

is a weak Haar subspace, then

$$\operatorname{span}\{G(h_1,t)(x), G(h_2,t)(x), \ldots, G(h_m,t)(x)\}$$

is a Haar subspace.

We again will make use of the tangent space.
Important: Switch between Gauß transform and linearization.

Theorem (W. 1998)

Let $F \subset C[a, b]$ be a set of functions of the form $f(p, x)$, depending **continously** on a parameter vector p.

Then

$$T(G(f(p, x))) = G(T(f(p, x)))$$

where as above T denotes the tangent space.

Algorithm for free knot splines (Outline)

- Problem: Given $f \in C[a,b]$, determine best approximation s^* of f w.r.t. $S_{m,n}$.

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Chacterization and uniqueness theorems, first numerical results: Meinardus and W. (2001)

Chebychev Approximation

Free Knot Splines

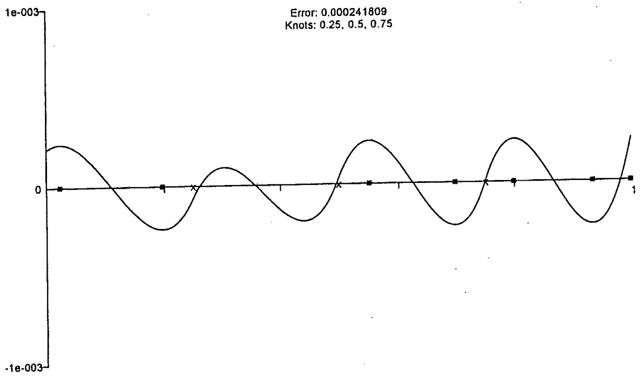
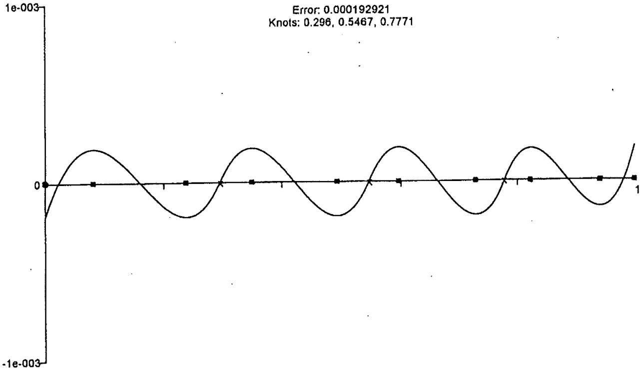


Figure: Approximation of $\exp(x)$: Fixed knots, $m = 3$, $k = 4$

Free Knot Splines



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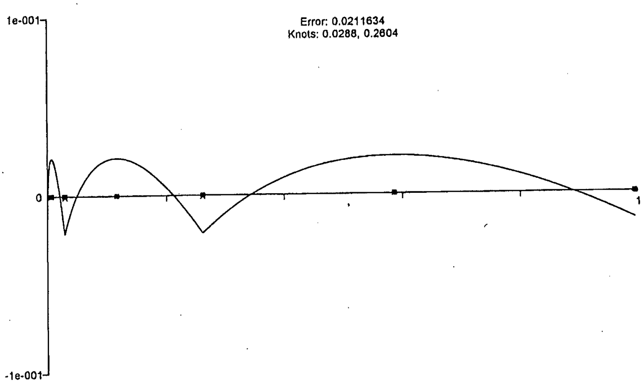

Figure: Approximation of $\sqrt{x}$: Free knots, $m = 2$, $k = 3$

Thank you very much for your attention

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Backup

Definition (Norm)

Given a vector space F. A **norm** $\|\cdot\|$ is a mapping from F into the real numbers with the following properties:

1. $\|f\| = 0 \Longrightarrow f = 0$
2. $\|a \cdot f\| = |a| \cdot \|f\|$ for all $f \in F$ and $a \in \mathbb{R}$
3. $\|f_1 + f_2\| \leq \|f_1\| + \|f_2\|$ for all $f_1, f_2 \in F$

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Properties

1. $\|f\| \geq 0$ for all $f \in F$
2. $\|f\| = 0 \Longleftrightarrow f = 0$

Strong uniqueness

Definition (Strong uniqueness)

Let V be a non-empty subset of $C[a,b]$ and $f \in C[a,b]$.
$v^* \in V$ is called **strongly unique best approximation** of f, if the exists a constant $\gamma_f > 0$, such that

$$\|f - v\| \geq \|f - v^*\| + \gamma_f \|v - v^*\|$$

for all $v \in V$.

Remark: Implies Uniqueness!

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L_2-Approximation

L_2-Approximation I

Let

$$\|h\|_2 = \left(\int_a^b (h(x))^2 dx\right)^{\frac{1}{2}}$$

denote the L_2-**Norm**, and set

$$L_2[a,b] = \left\{h : [a,b] \mapsto \mathbb{R}; \int_a^b (h(x))^2 dx < \infty\right\}.$$

This norm is induced by an **inner product**: Define, for $g_1, g_2 \in C[a,b]$,

$$\langle g_1, g_2 \rangle = \int_a^b g_1(x) \cdot g_2(x) dx$$

Then

$$\|h\|_2 = \langle h, h \rangle^{\frac{1}{2}}$$

L_2-Approximation II

Theorem (Existence and Uniqueness)

Let V be a finite-dimensional subspace of $L_2[a,b]$ and $f \in L_2[a,b]$. Then f possesses one and only one best L_2-approximation w.r.t. V.

Theorem (Characterization)

Let V be a finite-dimensional subspace of $L_2[a,b]$ and $f \in L_2[a,b]$. $v^* \in V$ is the best approximation of f w.r.t. V, if and only if

$$\langle f - v^*; v\rangle = 0 \text{ for all } v \in V.$$

L_2-Approximation III

v^* is the **orthogonal projection** of f onto V.

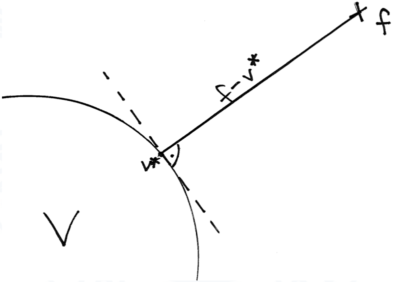

Figure: Orthogonal projection