Breaking the O(n m) Barrier for Buechi Games and Maximal End-Component Decomposition

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This Talk

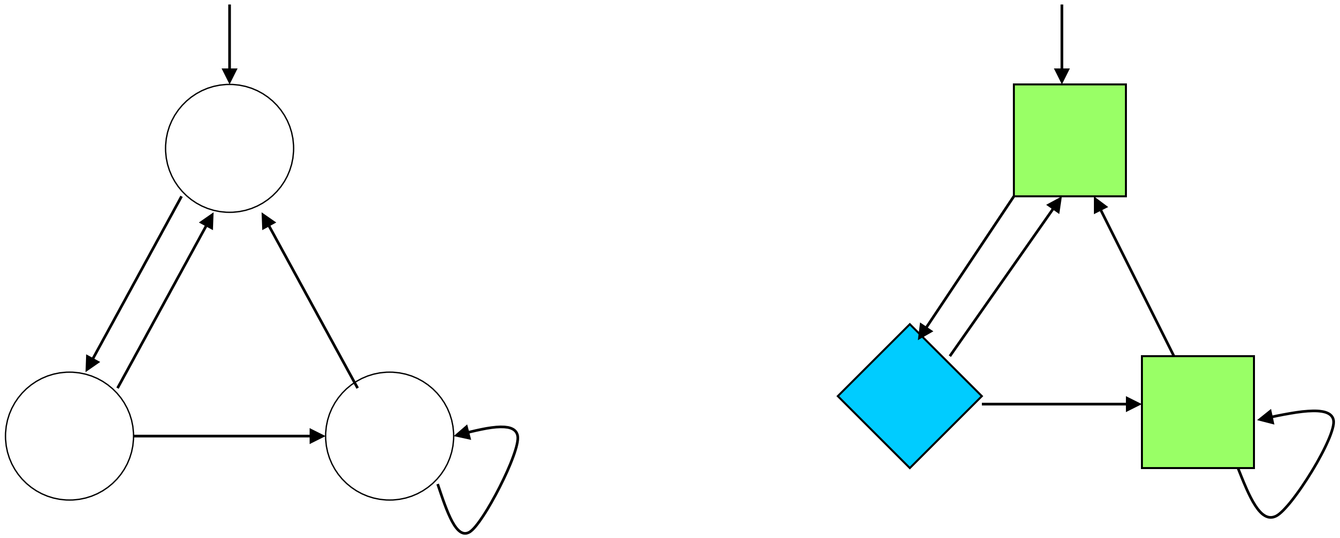
- Two classical algorithmic problems related to graph games and verification of probabilistic systems:
  - Buechi games.
  - Maximal end-component (MEC) decomposition.
- The long-standing best known bounds for these problems have been O(n m).
- This talk we will present algorithms for these problems to break the O(n m) barrier.

Motivation

- Roll back the clock
  - Tom Henzinger 2002: “very important and interesting algorithmic question”.
  - Orna Kupferman 2002: “very nice theoretical problem”.
  - Moshe Vardi 2022: talk on importance of Buechi automata and synthesis.

- Real motivation
  - Synthesis.
  - Model checking of open systems.
  - Probabilistic verification.

Graphs vs. Games



Two interacting players in games: Player 1 (Box) vs Player 2 (Diamond).

AND-OR Graphs.

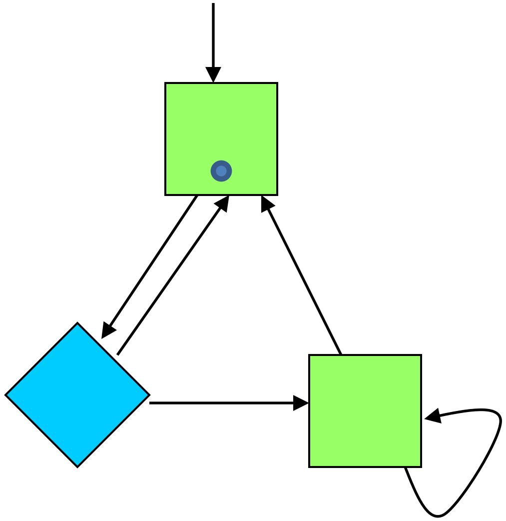
Game Graphs

Game Graphs

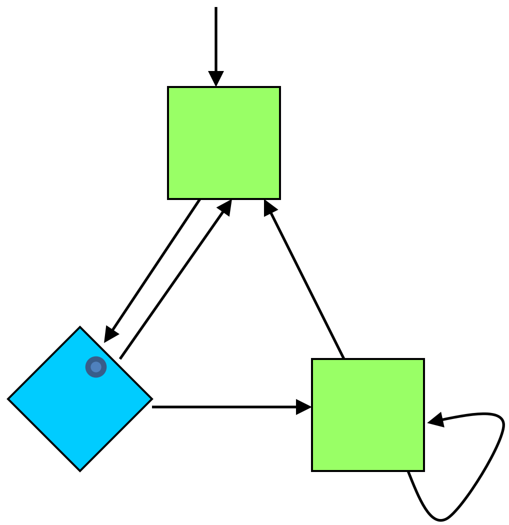
- A game graph $G= ((V,E), (V_1, V_2))$
  - Player 1 states (or vertices) V_1 and similarly player 2 states V_2, and (V_1, V_2) partitions V.
  - E is the set of edges.
  - E(v) out-going edges from v, and assume E(v) non-empty for all s.
  - Notation: $n= |V|$, $m =|E|$.

- Game played by moving tokens: when player 1 state, then player 1 chooses the out-going edge, and if player 2 state, player 2 chooses the outgoing edge.

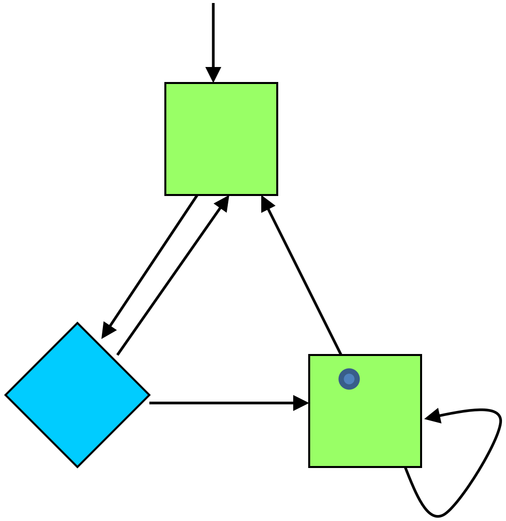
Game Example



Game Example



Game Example



Strategies

- Strategies are recipe how to move tokens or how to extend plays. Formally, given a history of play (or finite sequence of states), it chooses an out-going edge.
  - σ: $V^* \ V_1 \rightarrow V$.
  - π: $V^* \ V_2 \rightarrow V$.

Reachability and Buechi Objectives

- Reachability: there is a set of good vertices and goal is to reach them. Formally, for a set T of vertices, the objective is the set of infinite paths that visit the target T at least once.

- Buechi: there is a good set of vertices and goal is to visit them infinitely often. Formally, for a set B of vertices, the objective is the set of infinite paths that visit some vertex in B infinitely often. The objective is a liveness objective (like progress condition in mutual exclusion protocol).

Winning Set

- Winning set: Starting vertices such that player 1 has a strategy to ensure the objective against all strategies of player 2.

- Remark: Memoryless strategies are sufficient.

- We are interested in computing the winning set in games for player 1 for Buechi objectives.

Previous Result

- Reachability games:
  - O(m) (linear time algorithm) and PTIME-complete [Immerman 81, Beeri 80].
- Buechi games:
  - Classical algorithm: O(n m) [EJ91].
  - In the special case when m= O(n), an O(n^2/ log n) algorithm [CJH03]

Buechi Games Algorithm

Classical Algorithm

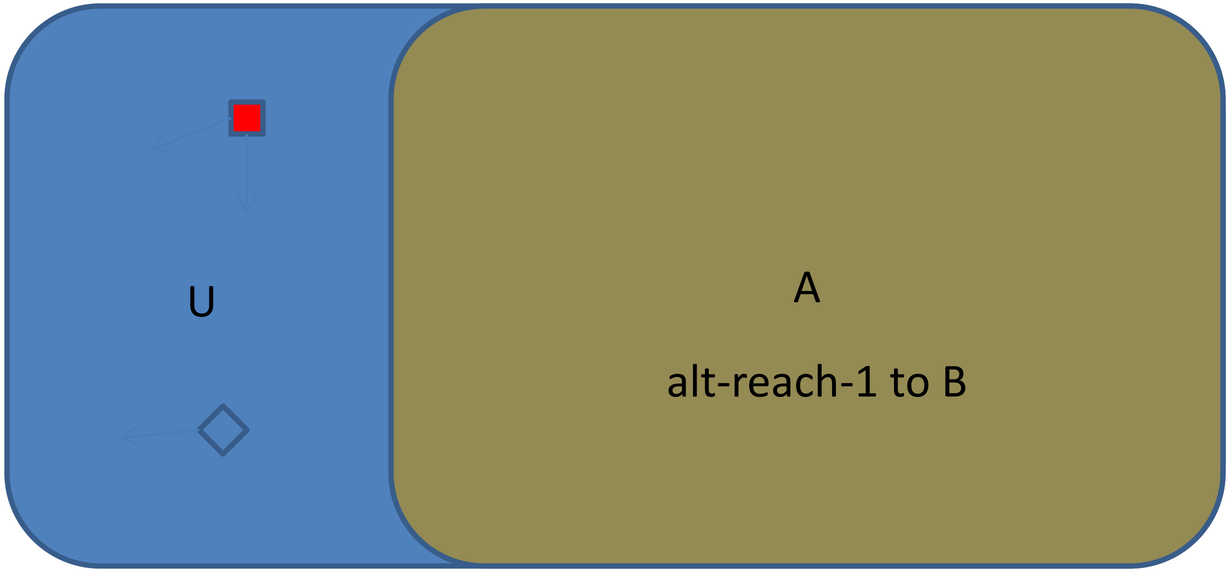
- A simple iterative algorithm using alternating reachability.

- Steps are as follows:
  1. Compute player-1 alt-reach set A to the current Buechi set.
  2. If A is the set of all vertices of current game graph, then stop and output A as the winning set.
  3. Else U be the remaining vertices (complement of A). Remove player-2 alt-reach set C to the set U from game graph and continue.

Classical Algorithm

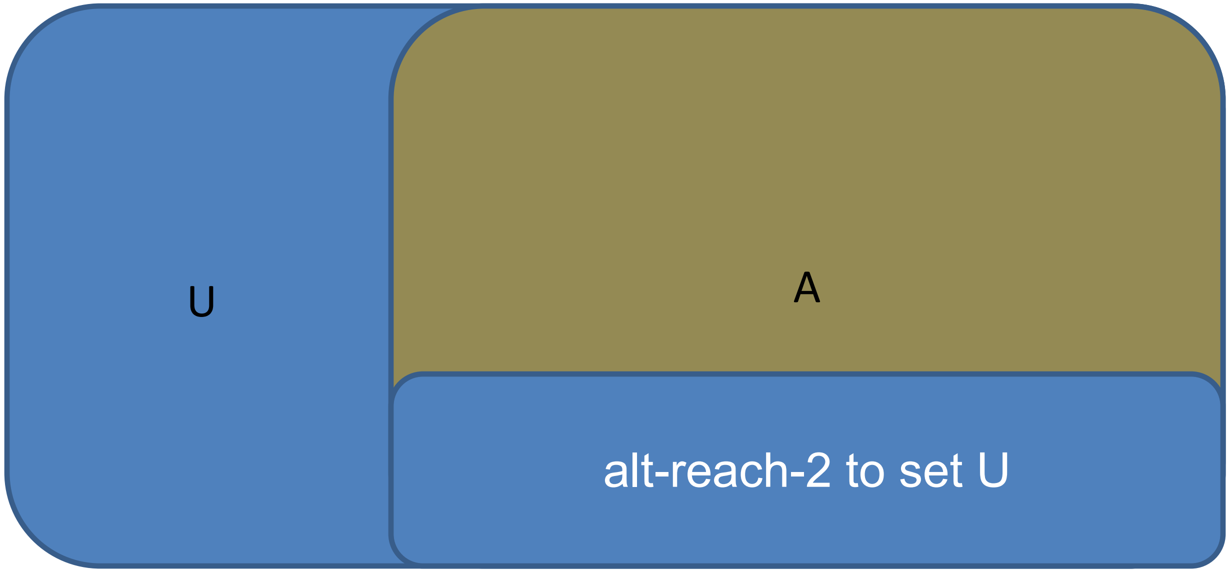
- Compute alt-reach for player 1 to the set B. Let us call this set A.

Classical Algorithm



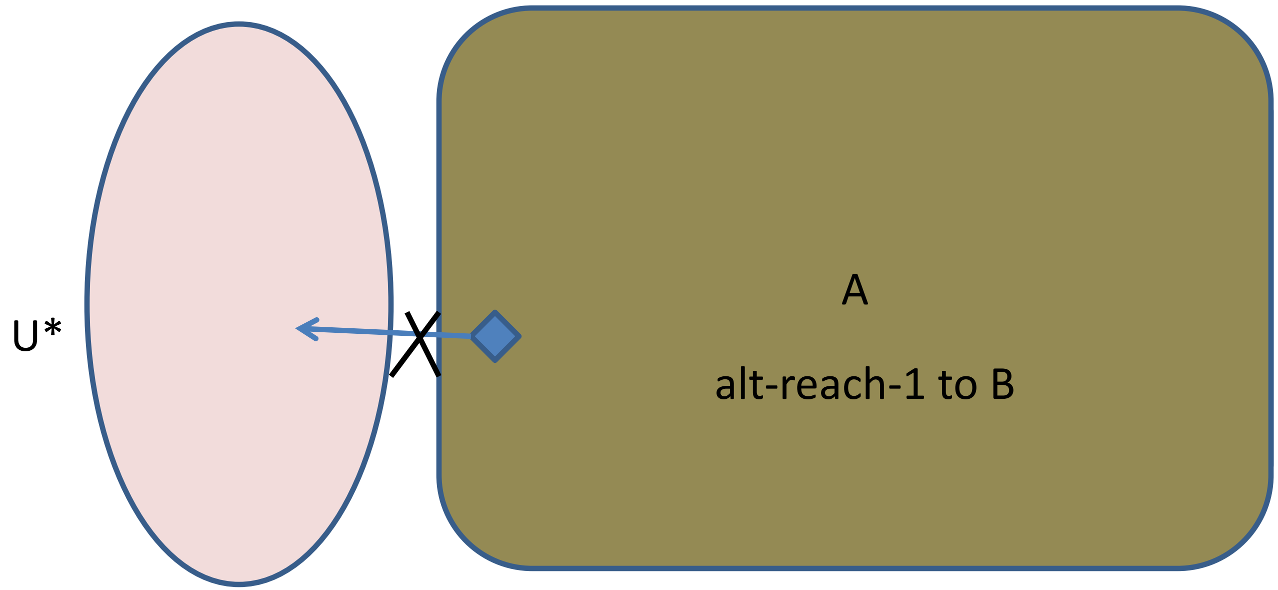
- Let U= V \ A. Then U is a trap. Clearly, U is not in winning for player 1.
- Hence alt-reach for player 2 to U is also not winning.

Classical Algorithm



- Iterate on the remaining sub-graph.
- Every iteration what is removed is not part of winning set.
- When the iteration stops, all remaining vertices are winning for player 1.

Correctness Proof Idea



- Player 2 cannot leave.
- Player 1 can ensure to reach, and again, and again and so on.

Classical Algorithm

- Classical algorithm identifies the largest trap and removes the trap.
- At most n iterations with time O(m) each.
- Analysis O(n m) is tight.
- Remark: Player-2 alt-reachability overall iterations is O(m) (edges worked on are removed from the graph).

Our New Algorithm

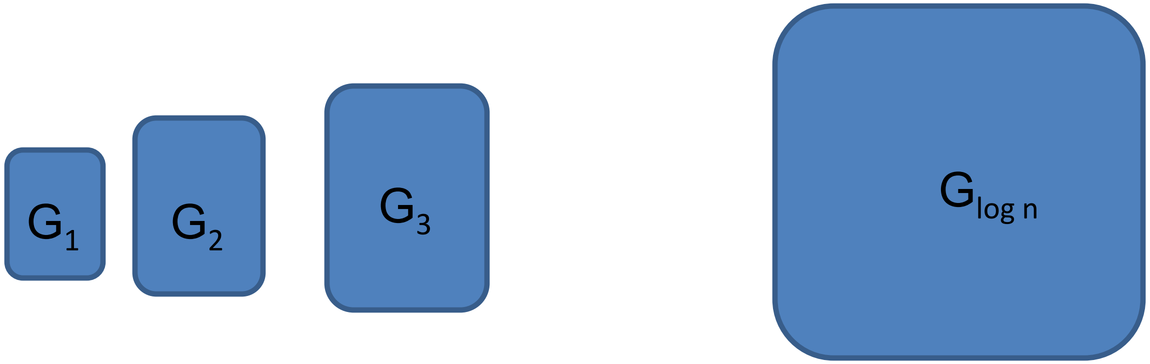
- Hierarchical graph decomposition technique.

- As long as we find traps, we can remove them, need not find the largest trap.

- Running time: $O(n^2)$
  - Better worst case for dense graphs.
  - Along with previous [CJH03] algorithm breaks O(n m) for all cases.

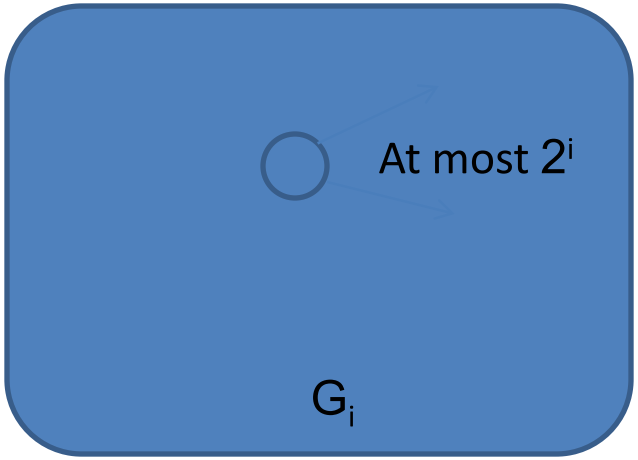
New Algorithm

- Create log n graphs hierarchically.
- Game graph $G_i=(V,E_i)$ ensuring $|E_i|$ is at most $O(n \cdot 2^i)$. Some special way to select edges according to ordering.



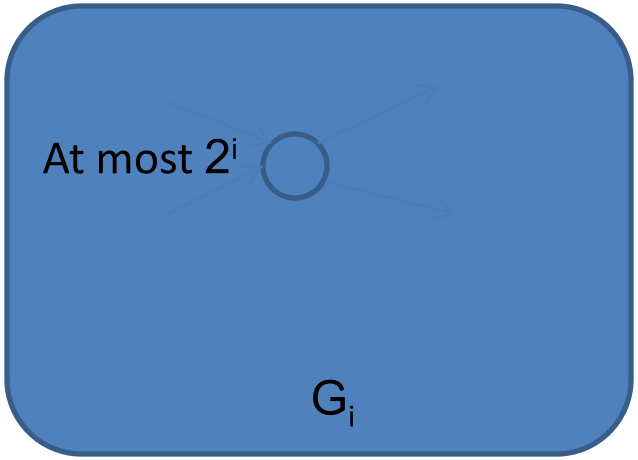
Construction of G_i

- The graph G_{i-1} is a sub-graph of G_i.
- In G_i, for every vertex add at most 2^i out-edges.



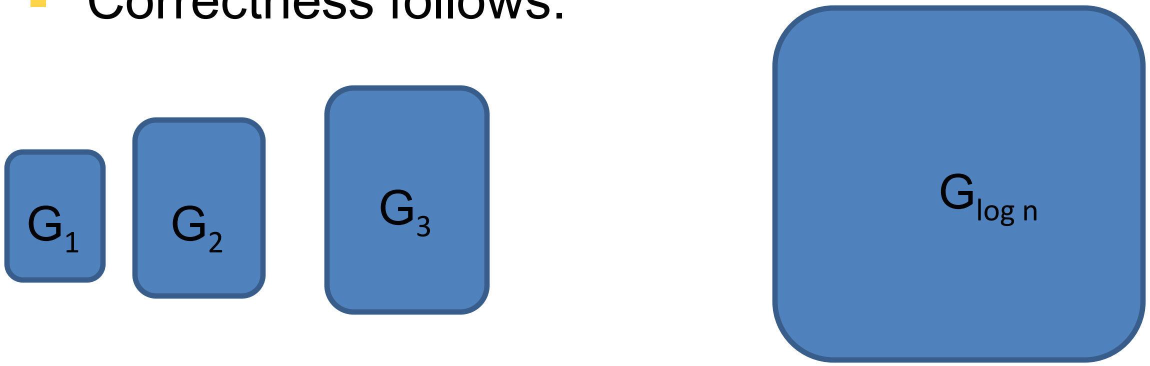
Construction of G_i

- The graph G_{i-1} is a sub-graph of G_i.
- In G_i, for every vertex add at most 2^i out-edges.
- Then for every vertex add at most 2^i more in-edges with preference to edges from player-2 non-Buechi vertices.



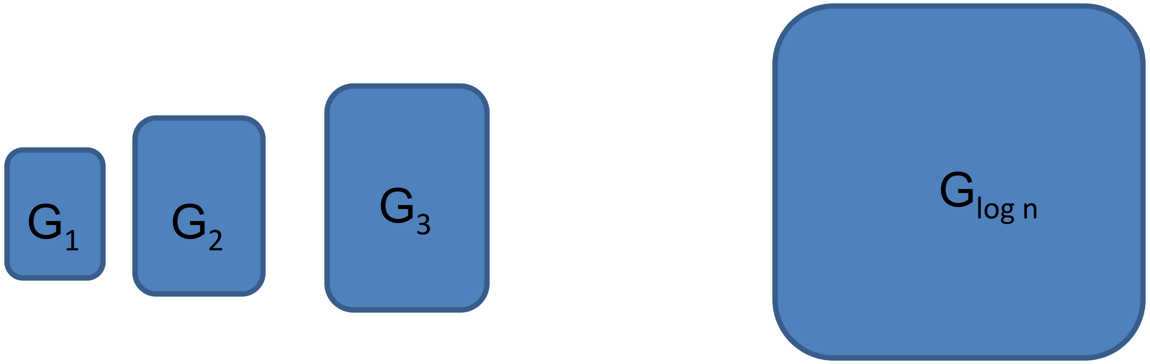
New Algorithm

- Search for traps in G_1, G_2, and so on.
  - Call a player-1 vertex with edges more than 2^i as blue in G_i.
  - Compute alt-1 reachability to the set of Buechi or blue vertices.
  - If the complement is non-empty, then that is a trap (some sense largest trap (without Buechi and blue) in G_i).
  - In the trap all player-1 edges are retained (any player-1 vertex where edges are not retained are blue and not in trap).
  - Correctness follows.



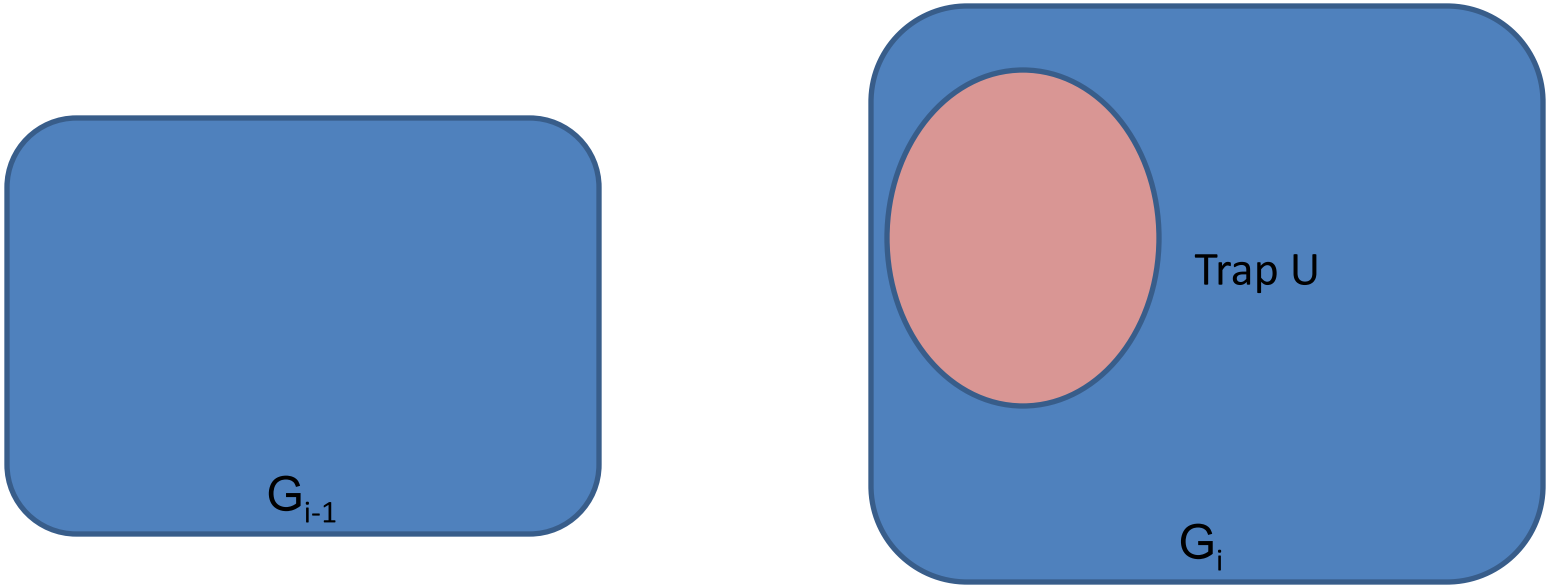
Correctness of New Algorithm

- Correctness is as follows:
  - When identify a trap, then all player-1 edges of the original game graph in the trap.
  - When algorithm stops no trap as in the final graph all edges are retained.
  - Correctness follows from classical algorithm.
- Challenge is running time analysis
  - We are working on more edges possibly (edge belong to several graphs).
  - Not clear we gain anything.
  - Challenge is to analyze the size of the trap we discover.



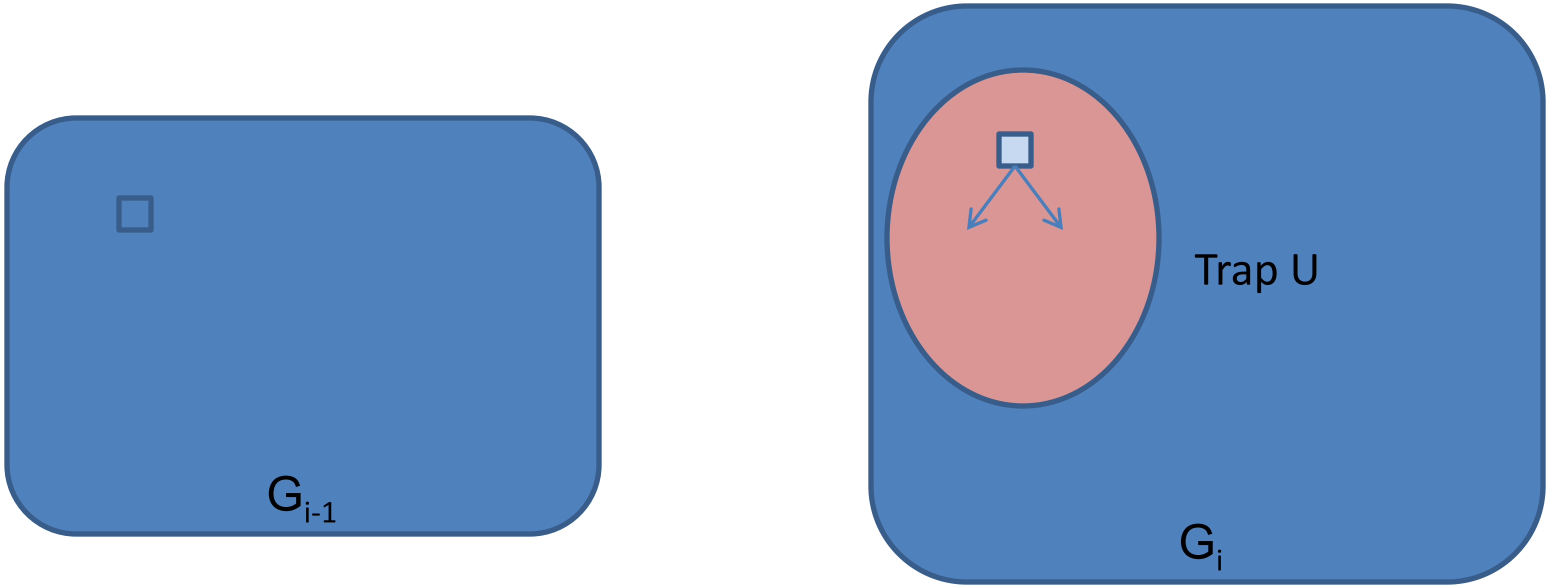
Running Time Analysis

- Analysis of the size of the trap.
  - Trap U identified in G_i but not in G_{i-1}.
  - We analyze the size of the trap we identify.



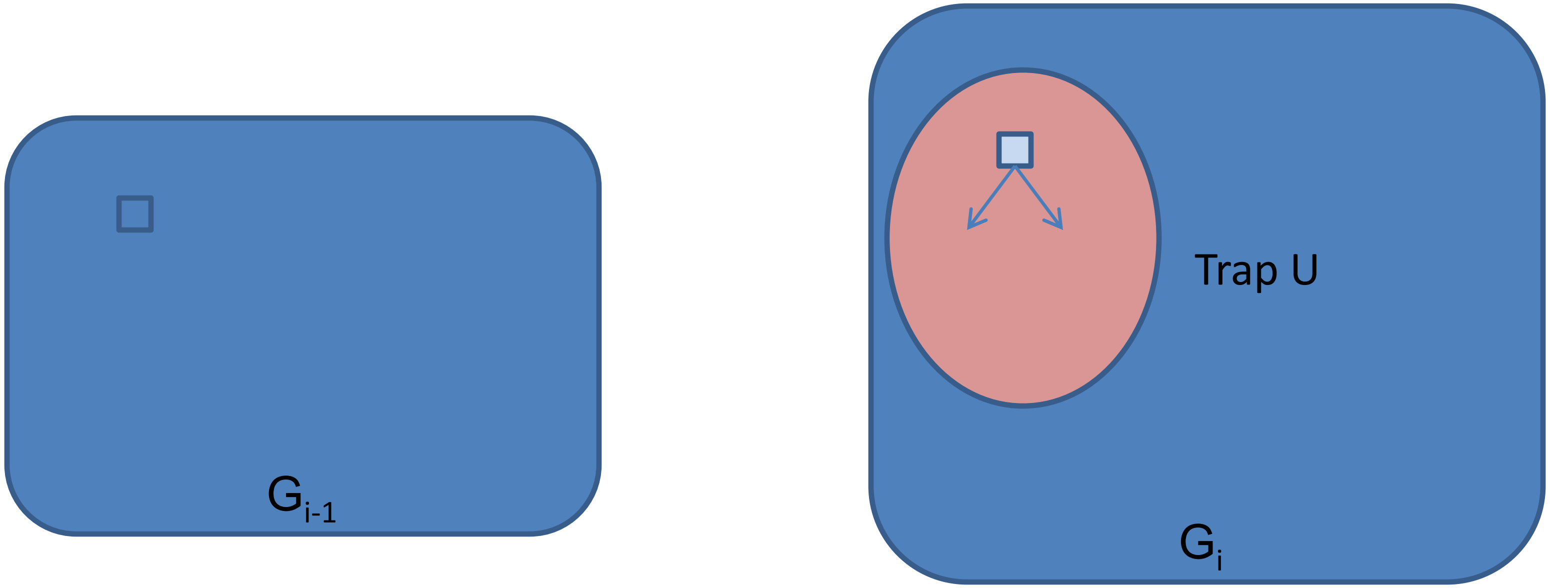
Running Time Analysis

- Analysis of the size of the trap.
  - Trap U identified in G_i but not in G_{i-1}.
  - Case 1: U contains a player-1 vertex v that was blue in G_{i-1}.
  - Then v has at least 2^{i-1} out-edges, otherwise would not have been blue.
  - Since a trap all out-going edges from v in trap. Size of trap at least 2^{i-1}.



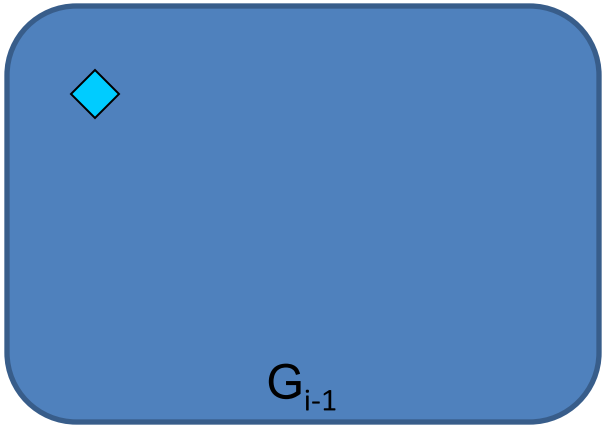
Running Time Analysis

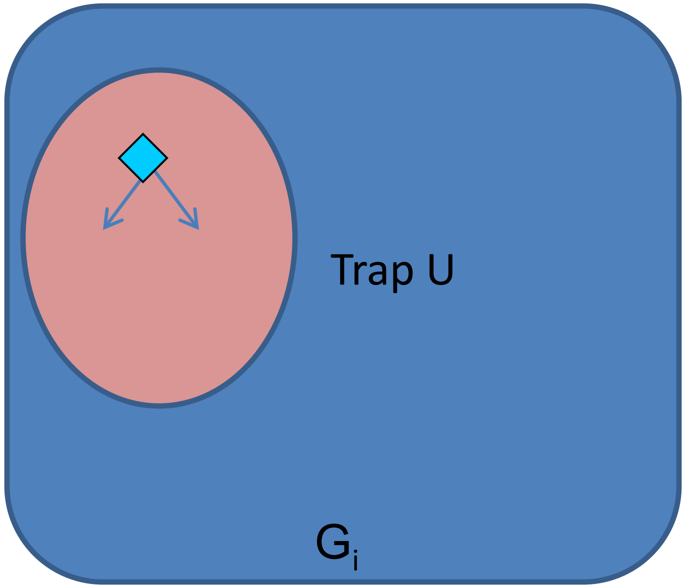
- Analysis of the size of the trap.
  - Trap U identified in G_i but not in G_{i-1}.
  - Case 1: Done. U does not contain a player-1 vertex v that was blue in G_{i-1}. All player-1 edges in G_i and G_{i-1} identical.
  - Case 2: Two sub-cases to analyze.



Running Time Analysis

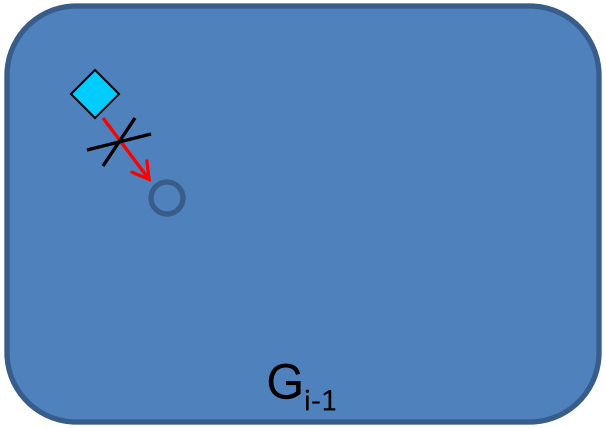
- Analysis of the size of the trap.
  - Case 1: All player-1 edges in G_i and G_{i-1} identical.
  - Case 2 (a): All player-2 edges in G_i and G_{i-1} are identical. Then U is a trap in G_{i-1} and this a contradiction.
  - Case 2(b): One new player-2 edge in the trap.

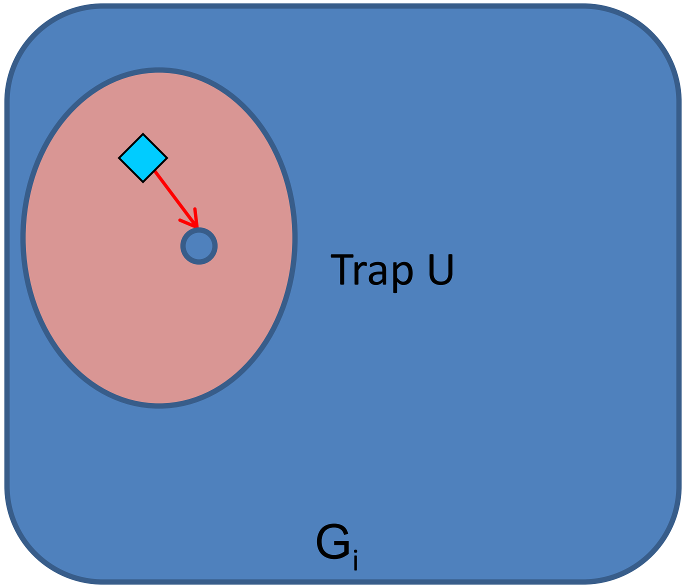




Running Time Analysis

- Analysis of the size of the trap.
  - Case 1: All player-1 edges in G_i and G_{i-1} identical.
  - Case 2 (a): All player-2 edges in G_i and G_{i-1} are identical.
  - Case 2(b): One new player-2 edge (u,v) in the trap.
  - Vertex v has at least 2^{i-1} in edges from player-2 non-Buechi vertices as they have the priority.





Running Time Analysis

- Analysis of the size of the trap.
  - Case 1: All player-1 edges in G_i and G_{i-1} identical.
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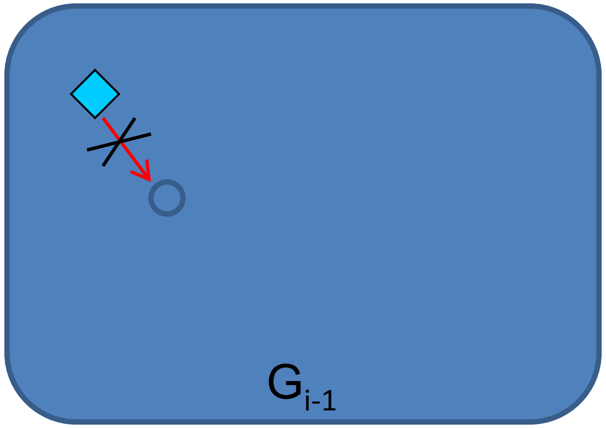
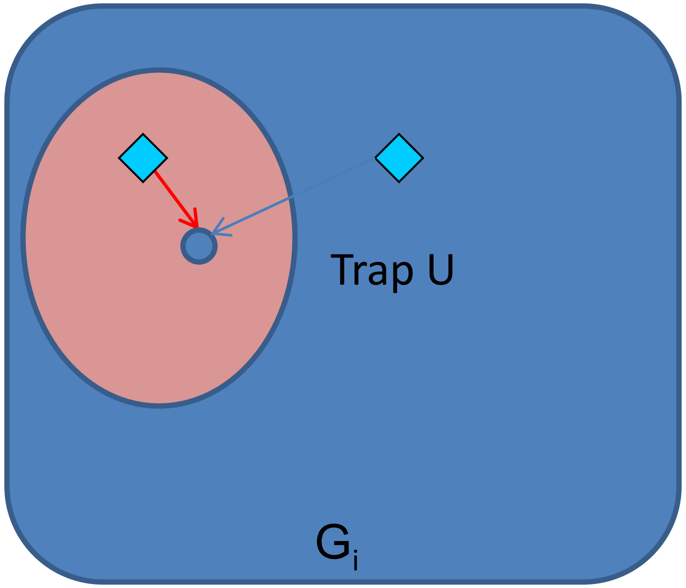
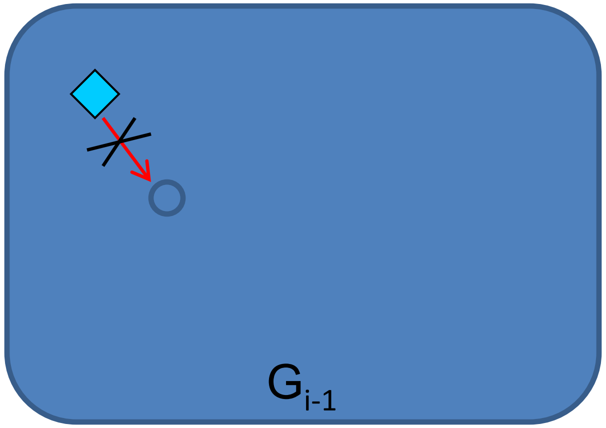
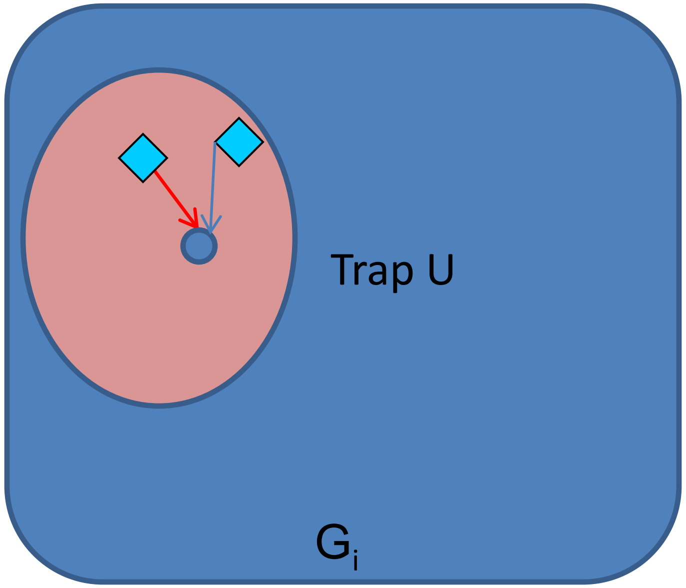




Running Time Analysis

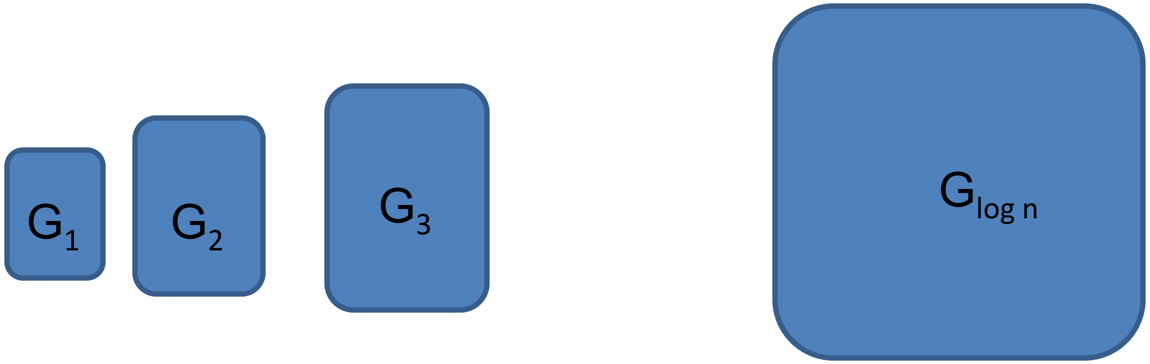
- Analysis of the size of the trap.
  - Case 1: All player-1 edges in G_i and G_{i-1} identical.
  - Case 2 (a): All player-2 edges in G_i and G_{i-1} are identical.
  - Case 2(b): One new player-2 edge (u,v) in the trap.
  - Vertex v has at least 2^{i-1} in edges from player-2 non-Buechi vertices as they have the priority.
  - All in-edges in the trap.
  - Size of trap at least 2^{i-1}.





New Algorithm

- Search for traps in G_1, G_2, and so on.
- If we find a trap, then remove it from all graphs.
- Key argument: if we find a trap in G_i, then size of trap is at least 2^{i-1}.
- Work done $O(n \cdot 2^{i+1})$ and charge to vertices removed.



Result

- New algorithm :$O(n^2)$.
  - Time spent to identify trap is $O(n \cdot 2^{i+1})$ and charge to the trap of size 2^{i-1}.
- Strikingly simple algorithm breaks the long standing $O(n\ m)$ barrier.
- Along with $O(n^2/\log n)$ algorithm for $m=O(n)$ of [CJH03] we break $O(n\ m)$ barrier for all cases.

Maximal End-component Decomposition

Maximal End-component Decomposition

- An end-component U is a set of vertices such that
  - Graph induced by U is strongly connected.
  - For all player-2 vertices in U all out-going edges end in U.
  - Typically used in MDPs (where player 2 is the probabilistic player).

  - We keep the notations uniform as our goal is to present algorithm for the problem.

- Maximal end-component (MEC) decomposition:
  - Classical algorithm: O(n m) [CY95, deAlfaro97]

Classical Algorithm

- A simple iterative algorithm using scc decomposition and alternating reachability.

- Steps are as follows:
  1. Compute the bottom scc's. They are all mec's and let their union be U.

  2. Remove player-2 alt-reach set C to the set U from game graph and continue.

  3. Stop when all vertices are removed.

Remark: Player-2 alt-reach over all iterations is O(m). Main work is repeated scc decomposition

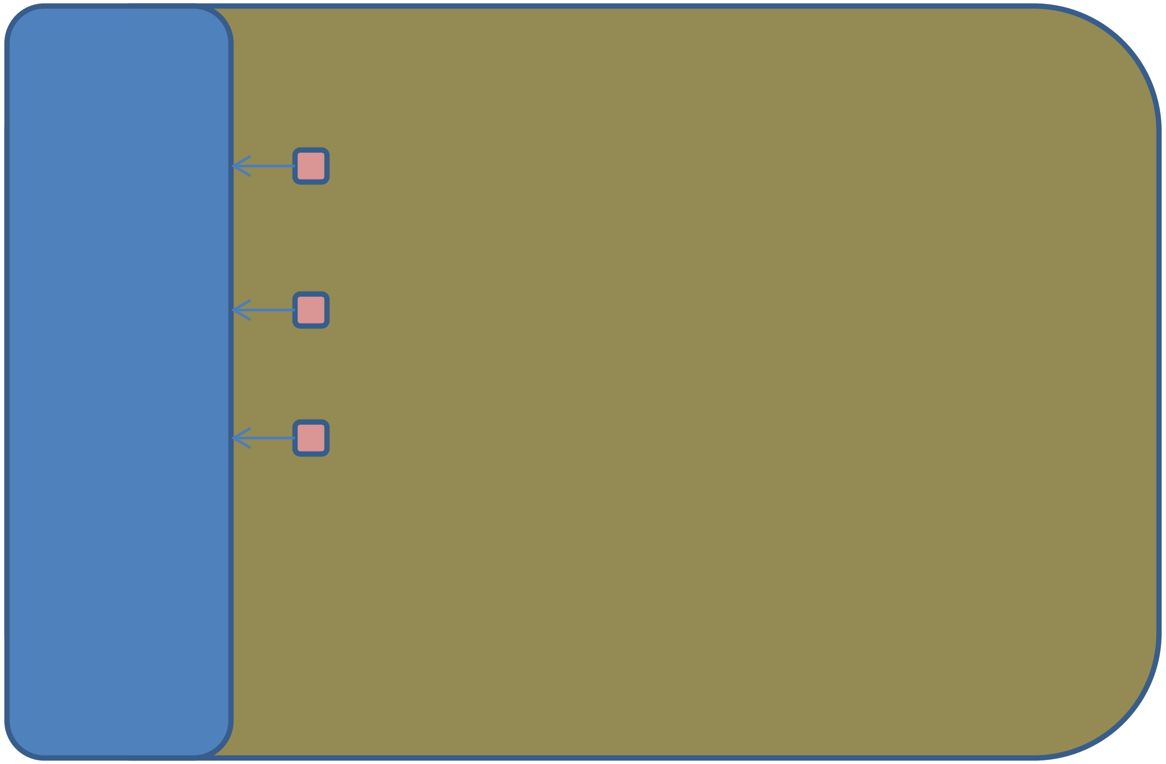
New Algorithm

- Same as for Buechi games using hierarchical graph decomposition technique.
- Instead of traps search for bottom scc's.
- $O(n^2)$ time algorithm.
- We also present a different improved algorithm.

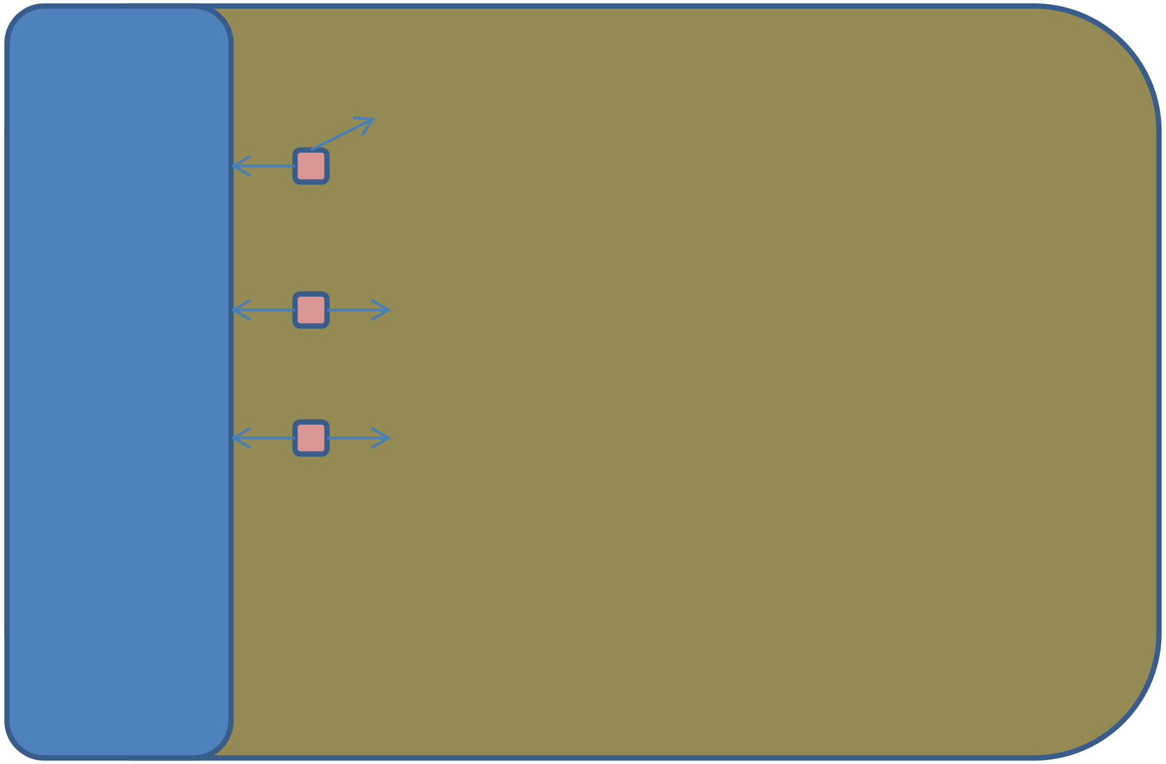
Different Sub-quadratic Algorithm

- The different sub-quadratic algorithm splits the classical algorithm as follows:
  - If more than $m^{0.5}$ edges were lost, then use classical scc decomposition algorithm.
  - If less than $m^{0.5}$ edges were lost, then use Tarjan scc algorithm with vertices having lost edges as starting point in lock-step.

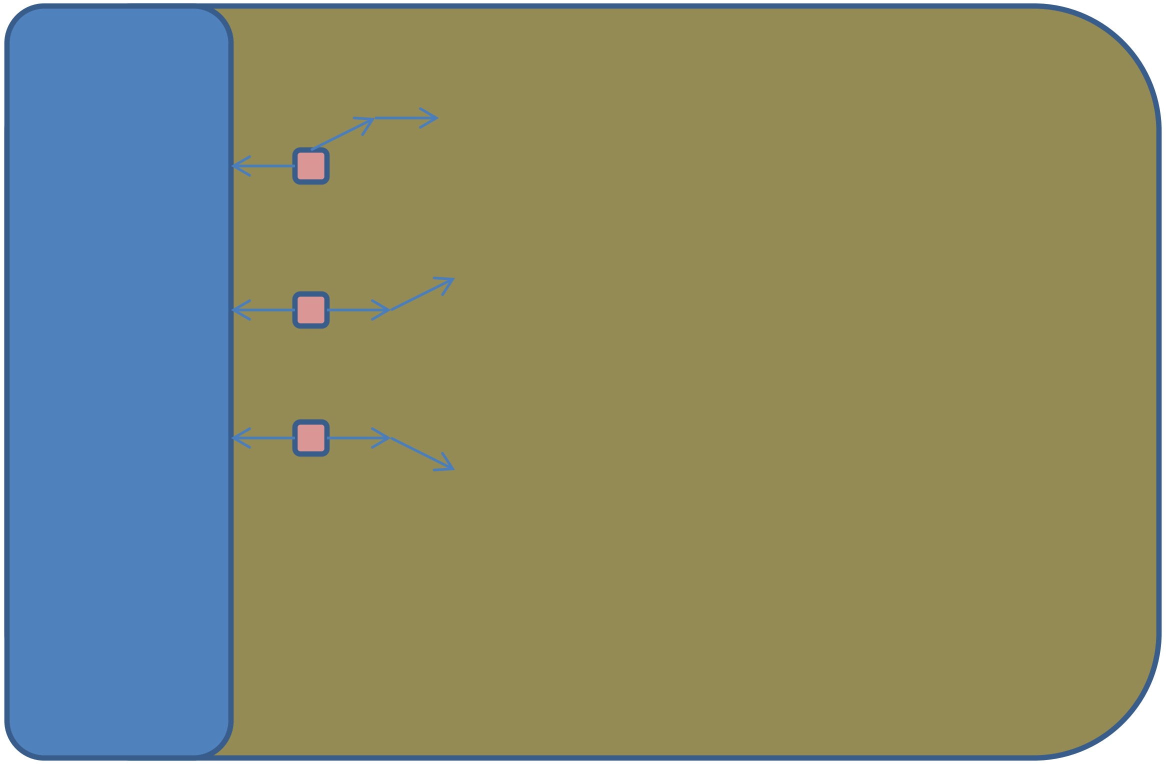
Different Sub-quadratic Algorithm



Different Sub-quadratic Algorithm



Different Sub-quadratic Algorithm



Stop when a bottom scc C is found. The bottom scc C is removed and work done is at most edges in C times $m^{0.5}$.

Different Sub-quadratic Algorithm

- The different sub-quadratic algorithm splits the classical algorithm as follows:
  - If more than $m^{0.5}$ edges were lost, then use classical scc decomposition algorithm.
    - Total work: $O(m^{1.5})$ since at most $m^{0.5}$ iterations of $O(m)$ time each.
  - If less than $m^{0.5}$ edges were lost, then use Tarjan scc algorithm with vertices having lost edges as starting point in lock-step.
    - Total work: $O(m^{1.5})$ since every removed edge is charged at most $O(m^{0.5})$.

MEC Decomposition

- Two algorithms:
  - $O(n^2)$.
  - $O(m^{1.5})$.
  - Clearly can do the min of the above two.

Summary

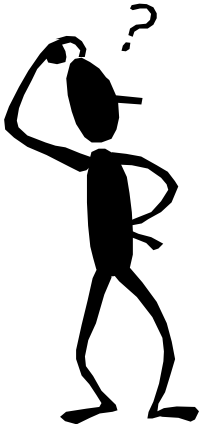
- Buechi games: a simple $O(n^2)$ time algorithm improving long-standing $O(n\ m)$ bound.
- MEC decomposition in time $O(\min(m^{1.5}, n^2))$ (worst case $O(m\ n^{2/3})$).

Conclusion

- Buechi games and MEC decomposition:
  - A core algorithmic problem in verification with long-standing $O(n\ m)$ barrier.
  - We present a simple $O(n^2)$ time algorithm for the problem, also for mec decomposition.
  - For mec decomposition also $O(m^{1.5})$ algorithm that gives a worst case $O(m\ n^{2/3})$ algorithm.
- Open questions:
  - $O(m\ n^{1-\epsilon})$ or $O(n\ m^{1-\epsilon})$ for Buechi games, for some $\epsilon > 0$.
  - $O(m\ n^{1/2})$ algorithm for mec decomposition.

The end

Thank you !



Questions ?